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EXPONENTIAL TIME DIFFERENCING SCHEME FOR MODELING THE DISSIPATIVE KAWAHARA SOLITONS IN A TWO-ELECTRONS COLLISIONAL PLASMA

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Abstract. In this work, a high efficiency and stable numerical method, which is called exponential time differencing fourth-order Runge-Kutta (EDT4RK) method, is introduced for the investigation of strong nonlinear and dispersive integrable and non-integrable differential equations. This method is applied for studying higher-order nonlinear structures that arise and propagate in plasma. The fluid equations of a plasma consisting of two electrons with different temperatures and immobile positive ions are reduced to a damped Kawahara equation *via* the reductive perturbation method. The damping term in the evolution equation appears due to taking the effect of electron-ion collision into account. If the damping term is ignored, the evolution equation reduces to the integrable Kawahara equation. The last equation has a hierarchy of exact solutions such as solitons and cnoidal waves. For modeling the dissipative solitons in the present plasma model, the numerical solution to the damped Kawahara equation is obtained *via* operating the EDT4RK method. The influence of the physical parameters related to the model under consideration on the higher-order dissipative solitons is investigated. The obtained results are novel solutions for non-integrable/damped Kawahara equation. The EDT4RK method is a powerful and a fourth-order accurate technique for obtaining numerical solutions with high accuracy and longtime stable.

Key words: exponential time differencing fourth-order Runge-Kutta method, damped Kawahara equation, collisional plasma, superthermal electrons.

1. INTRODUCTION

Modeling many physics and engineering problems results in differential, integral, and partial differential equations, which are either linear or nonlinear [1–20]. Therefore, those equations play a vital role in explaining and understanding the main characteristics of many natural phenomena and laboratory data. The key role of those equations has been extended to a series of fields such as industry, medicine, drug manufacturing, image processing, and electronic chip manufacturing.

The fifth-order Korteweg-de Vries (fKdV) equation, sometimes called Kawahara equation (KE), describes the propagation of different nonlinear structures such as solitons, cnoidal waves and others. These nonlinear waves can propagate in different physical settings such as plasma, shallow water with surface tension, chemical kinematics, astrophysics, and capillary-gravity water. The standard formula for the KE is given by [21]:

$$\partial_t \Psi + a\Psi\partial_x \Psi + b\partial_{3x}\Psi + R\Psi = 0, \tag{1}$$

where a, b, and R are functions of physical parameters. Equation (1) has been studied both numerically and analytically in literature. The analytical methods for solving the KE (1) are a new constraint among parameters that gives N-soliton solutions [22], an analytical method by using the improved hyperbolic and exponential ansatze for various solitons solutions [23], a Riemann-Hilbert method for soliton and breathers solutions [24], a Hirota's bilinear method with perturbation expansion, and a parametric limit method for lump solitons waves and for a variety of N-soliton forms [25, 26]. Moreover, it has been computed numerically by the improved variational iteration algorithm-I (mVIA-I) and algorithm-II [27, 28], Differential Quadrature Method [29], hybridizable discontinuous Galerkin technique [30], and by the geometric singular perturbation analysis whose solutions in solitary waves form [31]. All above-mentioned studies focused on the study of the integrable equation without taking the collisional effect into consideration. If the collisional impact due to the collisions between the charged particles and the neutral particles in a plasma model is taken into account, we finally get a non-integrable/damped Kawahara equation

$$\partial_t \Psi + a\Psi\partial_x \Psi + b\partial_{3x}\Psi + R\Psi - d\partial_{5x}\Psi = 0, \tag{2}$$

where $R = \mu/2$ and μ is the frequency of the ion-neutral collision. Equation (2) is a non-integrable Hamiltonian system, thus its integration cannot be performed analytically. Therefore, our goal in this article is to solve the non-integrable KE (2) numerically in order to study the effects of the dissipative nonlinear structures (*e.g.* dissipative solitons) propagating in different plasma models. To do that, the EDT4RK method has been chosen to analyze the damped KE (2) due to its high accuracy [32]. Moreover, a realistic physical application will be introduced with considering the collisional effect by the interaction between the charged particles themselves or between the charged and neutral particles or between the charged particles and plasma field.

The article is divided into the following Sections. In Sec. 2 we give a brief explanation of the ETD4RK method and the approximate error. In Sec. 3 we discuss the physical application in plasma physics of the damped KE and its numerically solution by the ETD4RK method. In Sec. 4 we give a summary of the work and of the obtained results.

2. EXPONENTIAL TIME DIFFERENCING METHOD

Various time integration methods that are designed to numerically approximate a solution for nonlinear partial differential equations have been developed [33]. However, explicit time integration methods are not handy for the reason that the time step is restricted to a limit on the size. On the other hand, utilizing implicit temporal methods implies solving large implicit systems at each time step. These types of methods are usually of low-order time step. In addition, the discretization of the space has been often done by using lower-order central finite differences schemes [34, 35]. These schemes are easy to implement with the boundary condition.

An alternative to the conventional methods is the Exponential Time Differencing (ETD) schemes [36, 37]. These schemes have arose in conjunction with spectral methods [38] that generate accurate spatial derivative using built-in codes in Matlab. The ETD schemes work as follows: for semi linear partial differential equations (PDEs), the linear part is integrated exactly, then the nonlinear part is approximated and then is integrated the approximation exactly.

ETD schemes of the Runge-Kutta (RK) type are obtained when approximating the non-linear terms by Runge-Kutta-like stages. The ETD methods' coefficients can be approximated by the Trapezium rule accurately [39] once prior to the integration for a fixed time step. The evaluation the the coefficients in the ETD4RK method requires the 'Cauchy integral' approach [40, 41]. In addition, the method needs to perform four function transforms per time step throughout the integration.

Assume we aim to find the solution of

$$\partial_t \Psi = N(\Psi) + L(\Psi),$$

where N and L are nonlinear and linear terms respectively. The first step is transferring the given equation to Fourier space with periodic boundary conditions, thus we obtain the following ordinary differential equation (ODE):

$$\frac{d\hat{\Psi}_k(t)}{dt} = L(\hat{\Psi}_k(t)) + \mathbf{fft}(N(\hat{\Psi}(t))), \tag{3}$$

where the fast Fourier transform (FFT) is represented with the Matlab command $\mathbf{fft}$.

The linear part $L(\hat{\Psi}(t))$ becomes a diagonal matrix with elements and the $N(\hat{\Psi}(t))$ is evaluated in physical space at the uniform grid points and then transformed back to spectral space by the Runge-Kutta method using the given initial condition. In the next Section, we will introduce the application of non-integrable equation and its numerical solution by the ETDRK4 method.

3. APPLICATION

3.1. DISSIPATIVE KAWAHARA SOLITONS IN A COLLISIONAL PLASMA

In this application, the fluid system of equations of a plasma model are built to study the interaction of electrostatic electron-acoustic dissipative solitons in a collisional unmagnetized plasma consisting of two different types of electrons (inertial cold electron and superthermal hot electron) and stationary positive ion. In addition, the collisional effect due to the collision between the cold electron and the neutral particles is taken into consideration. Accordingly, the dynamics of dissipative nonlinear electrostatic waves are governed by below dimensionless equations

$$\left\{\begin{array}{c} \partial_\tau n_c + \partial_\zeta (n_c u_c) = 0, \\ \partial_\tau u_c + u_c(\nu_{ei} + \partial_\zeta u_c) - \alpha \partial_\zeta \phi = 0, \\ \partial_{2\zeta}\phi - \frac{1}{\alpha} n_c - n_h + \beta = 0, \end{array}\right. \tag{4}$$

where n_c, n_h, and n_i indicate the normalized density of the cold electron, hot electron, and positive ion, respectively, u_c is the normalized fluid velocity of cold electron, ϕ is the normalized electrostatic wave potential, $\alpha = n_{h0}/n_{c0}$ and $\beta = Z_i n_{i0}/n_{h0} \equiv (1+1/\alpha)$ are the concentrations of hot electron and positive ion, respectively, where $n^{(0)}_{i0,c0,h0}$ represents the unperturbed density of the plasma species, Z_i shows the number of protons residing on the surface of the ion, and ν_{ei} denotes the normalized electron-ion collision frequency. The normalized hot electron density is given by the following superthermal distribution: $n_h = [1-\phi/(\kappa-3/2)]^{-\kappa+1/2}$, where κ indicates the spectral index.

The RPT is utilized for studying the dissipative nonlinear structures in the present model. According to this technique, the stretched of the independent variables is given by: $x = \sqrt{\varepsilon}(\zeta - \lambda\tau)$, $t = \sqrt{\varepsilon^3}\tau$, and $\nu_{ei} = \sqrt{\varepsilon^3}\nu$. In addition, the expansion of the dependent quantities reads: $n_c = 1 + \varepsilon n_{c1} + \varepsilon^2 n_{c2} + \cdots$, $u_c = \varepsilon u_{c1} + \varepsilon^2 u_{c2} + \cdots$, and $\phi = \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots$. By following the same procedure as in Ref. [42] and using the mentioned stretched and expansion of the (in)dependent variables into the basic equations of the model, we finally get the damped KdV equation (1) for $\Psi \equiv \phi_1$ with

$$\left\{\begin{array}{l} a = -b\left(\frac{2\alpha}{\lambda^4} + \frac{k_1 k_2}{k_3^2}\right), \\ b = \frac{\lambda^3}{2} \,\&\, \lambda = \sqrt{\frac{k_3}{k_1}}, \end{array}\right.$$

where $R = \nu_{ei}/2$, $k_1 = (\kappa - 1/2)$, $k_2 = (\kappa + 1/2)$, and $k_3 = (\kappa - 3/2)$.

Also, in the auroral plasma, the electrostatic high frequency noises could be excited due to the EAW and if these noises are taken into the cold electron momentum equation [43], we finally obtain Eq. (2), where $d << 1$. The soliton solution of Eq. (2) without damping term ($R = 0$) reads [43]:

$$\Phi = \Phi_{\max} \operatorname{sech}^4 \left[\frac{1}{W} \left(x - \frac{36b^2}{169d} t \right) \right], \tag{5}$$

where $\Phi_{\max} = \left(105b^2/169ad\right)$ and $W = \sqrt{52d/b}$ are, respectively, the amplitude and width of the nondissipative Kawahara soliton.

3.2. NUMERICAL SIMULATIONS

The aim of this Section is to present the efficiency of the ETD-RK methods [44], and conduct numerical studies on the model problem, namely Eq. (2), in one space dimension.

For the simulation tests, we use Fourier spectral methods to discretize the spatial derivatives. Hence, in Fourier space, the transformed KE (2), with periodic boundary conditions, is given by the system of ODEs in time t

$$\frac{d\hat{\Psi}_k(t)}{dt} = i(bk^3 + dk^5 + ir)\hat{\Psi}_k(t) - \frac{aik}{2}\mathbf{fft}(\hat{\Psi}(t)^2), \tag{6}$$

where $c = i(bk^3 + dk^5 + ir)$. We use the soliton solution

$$\Psi(x,t) = \frac{105b^2}{169ad} \operatorname{sech}^4 \left[\frac{1}{2}\sqrt{\frac{b}{13d}} \left(x - \frac{36b^2}{169d} t \right) \right],\ x \in [-5\pi, 5\pi], \tag{7}$$

as an initial condition for the KE (2).

For the temporal integration of the system of ODEs (6), we use the ETD4RK method

$$\begin{aligned}
\mathcal{A}_n &= \hat{\Psi}_n e^{d\Delta t/2} + (e^{d\Delta t/2} - 1)\mathcal{G}_n/d,\\
\mathcal{B}_n &= \hat{\Psi}_n e^{d\Delta t/2} + (e^{d\Delta t/2} - 1)\mathcal{G}(A_n, t_n + \Delta t/2)/d,\\
\mathcal{C}_n &= \mathcal{A}_n e^{d\Delta t/2} + (e^{d\Delta t/2} - 1)(2\mathcal{G}(\mathcal{B}_n, t_n + \Delta t/2) - \mathcal{G}_n)/d,\\
\hat{\Psi}_{n+1} &= \hat{\Psi}_n e^{d\Delta t} + \{((d^2\Delta t^2 - 3d\Delta t + 4)e^{d\Delta t} - d\Delta t - 4)\mathcal{G}_n\\
&\quad +2((d\Delta t - 2)e^{d\Delta t} + d\Delta t + 2)(\mathcal{G}(A_n, t_n + \Delta t/2) + \mathcal{G}(\mathcal{B}_n, t_n + \Delta t/2))\\
&\quad +((-d\Delta t + 4)e^{d\Delta t} - d^2\Delta t^2 - 3d\Delta t - 4)\mathcal{G}(\mathcal{C}_n, t_n + \Delta t)\}/(d^3\Delta t^2).
\end{aligned}$$

Here, d is the linear part of the system that is represented by a diagonal matrix, and $\mathcal{G}$ denotes the action of the nonlinear operator on $\hat{\Psi}$ on the grid, Δt denotes the time step size and $\hat{\Psi}_n$ and F_n represent the numerical approximation to $\hat{\Psi}(t_n)$ and $F(\hat{\Psi}(t_n), t_n)$, respectively.

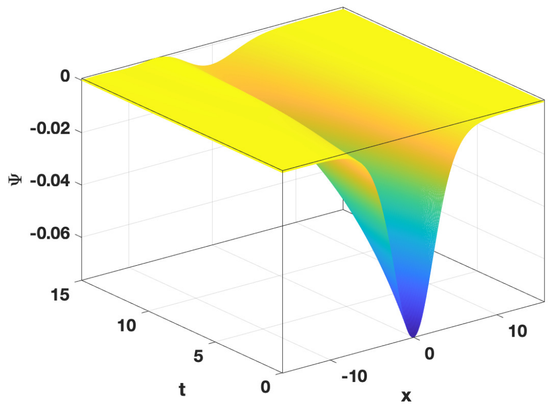

Fig. 1 – The profile of the dissipative Kawahara soliton according to the numerical solution is plotted in the (x,t) −plane for $(\kappa,\alpha,\nu_{ei}) = (3,1.8,0.3)$.

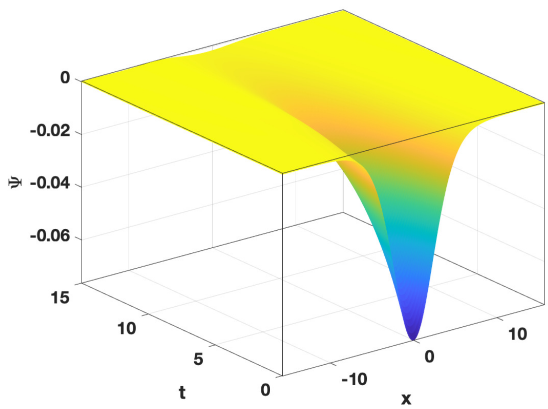

Fig. 2 – The profile of the dissipative Kawahara soliton according to the numerical solution is plotted in the (x,ν_{ei}) −plane for $(\kappa,\alpha) = (3,1.8)$.

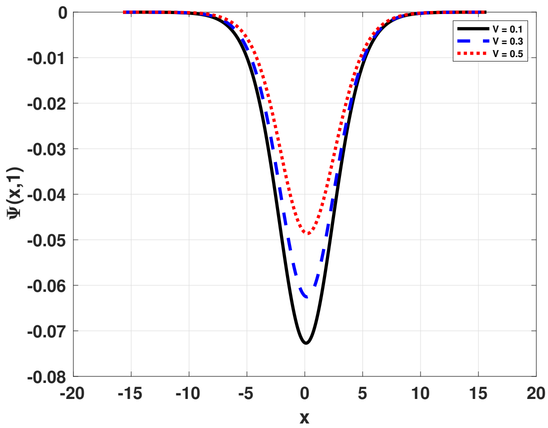

Fig. 3 – The profile of the dissipative Kawahara soliton according to the numerical solution is plotted against x with different values of the concentration of hot electron α. Here, $(\kappa, \nu_{ei}) = (3, 0.3)$.

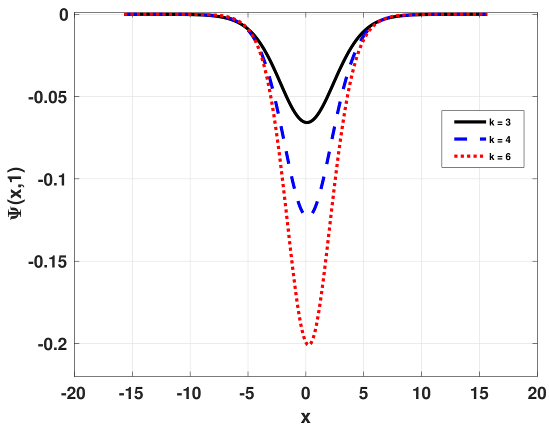

Fig. 4 – The profile of the dissipative Kawahara soliton according to the numerical solution is plotted against x with different values of the spectral index κ. Here, $(\alpha, \nu_{ei}) = (1.8, 0.3)$.

Now, for investigating the effect of relevant physical plasma parameters of the profile of the dissipative Kawahara solitons, the numerical solutions are plotted in Figs. 1-4.

Figures 1 and 2 demonstrate that the dissipative Kawahara soliton amplitude and width are decaying with increasing both time and collisional frequency ν_{ei}. Also, increasing the concentration of hot electron α leads to the reduction of the dissipative Kawahara soliton amplitude and width as shown in Fig. 3. The effect of spectral index of hot electrons κ is depicted in Fig. 4.

It is noted that by increasing the value of the spectral index κ, both the amplitude and width of the dissipative Kawahara solitons are increasing. This means that when plasma model is going to the Maxwellian state, the nonlinearity and dispersion are increased, which lead to the enhancement of both the amplitude and the width.

4. CONCLUSION

The aim of this work has been to present a first numerical solution of the linear damped Kawahara equation (non-integrable equation) by the ETD4RK method. The paper has introduced a physical application in plasma media, which describes the dissipative Kawahara solitons in a collisional two electrons plasma having cold inertial electron, hot inertialess superthermal electrons, and stationary positive ions. The effect of relevant physical plasma parameters, namely, the spectral index, the concentration of hot electrons, and the electron-ion collisional frequency on the profile of the dissipative Kawahara solitons has been examined numerically. The accuracy of the obtained solution has been examined by estimating the relative error when the ETD4RK numerical method is used. The obtained solution helps us to depict the dynamics of several phenomena and applications in mechanics, biology, engineering, and physics.

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