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OPTICAL SOLITONS IN SYSTEMS OF TWO-LEVEL ATOMS

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Abstract. A scientific-methodical review of the derivation of nonlinear evolution equations describing the interaction of laser pulses with a system of two-level atoms and having solutions in the form of optical solitons is presented. Regimes of propagation of resonant and quasi-resonant envelope solitons, as well as few-optical-cycle solitons with temporal durations from nano- to femtoseconds, are considered. The review is a short travel guide to selected problems of soliton propagation in media consisting of two-level atoms.

Key words: optical soliton, envelope, few-cycle pulse, two-level atom, nonlinearity, dispersion, resonance, quasi-resonance, integrability.

1. INTRODUCTION

Optical soliton is a solitary laser pulse of a certain duration (from nano- to femtoseconds) that has a carrier frequency in the visible or near infrared range and is able to propagate in a nonlinear dispersive medium without changing shape over long distances. It is also important that the solitons have the property of elastic interaction with each other. That is, after a collision, the solitons restore their original form. All this occurs in a nonlinear medium, therefore the superposition principle, as it is understood in linear media, is invalid. In connection with this property, solitons are placed high hopes for their use in optical communication systems. With the shortening of the duration of the soliton, the capacity of the corresponding information systems increases.

From a mathematical point of view, a soliton is a solution of a nonlinear equation (or system of equations) in partial derivatives. The equation under consideration here is integrable, *i.e.* using analytical approaches, one can find a solution of the corresponding Cauchy problem or the boundary value problem. The property of the elastic interaction of solitons with one another is due to the integrability of the equation in question. In addition to nonlinearity, the dispersion must take place for the existence of a soliton. When the light field interacts with atoms, this means that there is a temporary delay in the polarization response of the medium to the field effect. Mutual compensation of the nonlinear steepening of the profile of the wave packet and its dispersive spreading leads to the formation of a soliton.

A two-level atom is the simplest quantum model used in many problems, where the interaction of light with matter is considered. This model became particularly popular in the 60s of the last century, after the invention of lasers. If the frequency of light ω is close to the frequency ω_0 of the transition between any two quantum levels in the atom (the resonance case), then it is possible to restrict ourselves to a good accuracy by considering these two levels [1].

In the interaction of a two-level atom with a short light pulse, in general, the populations of quantum levels change, which affects the polarization response of the medium. The level populations are determined by the energy stored in the atom. The change in this energy must be accompanied by a change in the energy of the field of the light pulse. The energy of the pulse, in turn, is proportional to the square of the electric field. Thus, the change in the populations of the quantum levels is an extremely nonlinear effect with respect to pulse electric field.

It is clear that the shorter the pulse duration, the more clearly the delay in response of a two-level atom to the effect of a given pulse should be manifested. This temporal nonlocal connection between the polarization response $\mathbf{P}$ of a medium of two-level atoms and pulse electric field $\mathbf{E}$ is the dispersion, the presence of which, along with the above nonlinearity, is necessary for the formation of an optical soliton.

The purpose of this work is to conduct a scientific and methodical review of the physical conditions that generate various optical solitons in a system of two-level atoms. The main emphasis is on the derivation of the corresponding nonlinear wave equations and a brief description of the properties of their soliton solutions. The use of different approximations is also discussed in detail. After the derivation of such equations, their investigation of integrability is not carried out, but the very fact of integrability with references to original works or monographs is simply stated. The point is that the question of integrability itself is complex enough and it is hardly advisable to discuss it rigorously in this review. Here, first of all, the goal of physical analysis of the validity conditions of certain approximations, which lead to equations generating solitons, is pursued.

The paper is organized as follows. In Sec. 2 we derive a system of material and wave equations describing the interaction of electromagnetic pulses with a system of two-level atoms. In Section 3, the derivation of the Maxwell-Bloch system is represented by the approximation of slowly varying envelope (SVE). In Sec. 4, on the basis of resonant and quasi-resonant approximations, various nonlinear wave equations for the pulse envelope are derived from this system. Section 5 presents various situations when the SVE approximation is not applicable. On the basis of various approximations, nonlinear wave equations that generate soliton solutions are derived. Section 6 discusses the interaction of long and short waves in an anisotropic medium of two-level atoms. It is shown that this interaction has a direct relation to the optical method of generating the terahertz radiation. Section 7 contains the main conclusions of this paper.

2. WAVE AND MATERIAL EQUATIONS

For a theoretical study of the interaction of high-power light pulses with matter, a well-established semiclassical approach is commonly used: the electromagnetic pulse field is described by Maxwell's equations, and the response of matter to the effect of pulse field is given by the equations of quantum mechanics.

Let the light pulse propagate along the z-axis. The Maxwell equations easily yield a wave equation of the form

$$\frac{\partial^2 \mathbf{E}}{\partial z^2} - \frac{n_m^2}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2 \mathbf{P}}{\partial t^2}, \tag{1}$$

where c is the speed of light in vacuum, n_m is the refractive index of an isotropic matrix into which two-level atoms are placed, and t is the time.
Now we turn to the derivation of material equations using the quantum-mechanical approach.

The Hamiltonian of the atom is written as follows

$$\hat{H} = \hat{H}_0 + \hat{V}_{\text{int}}, \tag{2}$$

where $\hat{H}_0$ is the Hamiltonian of the electron interacting with the atomic core, $\hat{V}_{\text{int}}$ is the Hamiltonian of the interaction of the atom with the field of the external light pulse.

As a basis we will use the eigenfunctions $|1\rangle$ and $|2\rangle$ of operator $\hat{H}_0$:

$$\hat{H}_0|1\rangle = \varepsilon_1|1\rangle, \;\; \hat{H}_0|2\rangle = \varepsilon_2|2\rangle, \tag{3}$$

where ε_1 and ε_2 are the energies of the ground and excited states, respectively. Therefore,

$$\hat{H}_0 = \begin{pmatrix} \varepsilon_2 & 0 \\ 0 & \varepsilon_1 \end{pmatrix}. \tag{4}$$

Now we find the matrix form of the operator. We shall use the classical interaction Hamiltonian in the electro-dipole approximation:

$$\hat{V}_{\text{int}} = -\hat{\mathbf{d}}\mathbf{E}, \tag{5}$$

where $\hat{\mathbf{d}}$ is the operator of atomic dipole moment. Let us consider the matrix elements $\langle j|\hat{V}_{\text{int}}|k\rangle = -\langle j|\mathbf{dE}|k\rangle$. The wave functions of atomic electrons are localized on a scale length of the order of the Bohr radius $a_B \sim 10^{-8}$ cm, and the characteristic scale of the field inhomogeneity is the wavelength λ. For the visible spectral range we have $\lambda \sim 10^{-4}$ cm. This means that on the atomic scale length the field $\mathbf{E}$ can be assumed to be homogeneous with good accuracy. Then we write approximately: $\langle j|\mathbf{dE}|k\rangle \approx \langle j|\mathbf{d}|k\rangle\mathbf{E} = \mathbf{d}_{jk}\mathbf{E}$. Since the matrix $\hat{\mathbf{d}}$ is Hermitian, then $\mathbf{d}_{22}$ and $\mathbf{d}_{11}$ are real. Besides, $\mathbf{d}_{12} = \mathbf{d}_{21}^{*}$. In what follows we assume that $\mathbf{d}_{12} = \mathbf{d}_{21} = \mathbf{d} \in R$. As a result we have the expression (5), where

$$\hat{\mathbf{d}} = \begin{pmatrix} \mathbf{d}_{22} & \mathbf{d} \\ \mathbf{d} & \mathbf{d}_{11} \end{pmatrix}. \tag{6}$$

The density matrix $\hat{\rho}$ of a two-level atom can be represented in the form

$$\hat{\rho} = \begin{pmatrix} \rho_{22} & \rho_{21} \\ \rho_{12} & \rho_{11} \end{pmatrix}. \tag{7}$$

We introduce the real variables

$$U = \frac{\rho_{12} + \rho_{21}}{2}, \quad V = \frac{\rho_{12} - \rho_{21}}{2i}, \quad W = \frac{\rho_{22} - \rho_{11}}{2}, \tag{8}$$

taking into account the conservation law of the total population of atomic levels

$$\rho_{11} + \rho_{22} = 1. \tag{9}$$

Using von Neumann's equation

$$i\hbar\frac{\partial\hat{\rho}}{\partial t} = \left[\hat{H}_0 + \hat{V}_{\text{int}}, \hat{\rho}\right], \tag{10}$$

where $\hbar$ is the Planck constant, as well as expressions (4)–(9), we arrive at a system of the form

$$\frac{\partial U}{\partial t} = -(\omega_0 - \Omega_e)V, \tag{11}$$

$$\frac{\partial V}{\partial t}=\left(\omega_0-\Omega_e\right)U+\Omega_o W\,, \tag{12}$$

$$\frac{\partial W}{\partial t}=-\Omega_o V\,. \tag{13}$$

Here $\omega_0=\left(\varepsilon_2-\varepsilon_1\right)/\hbar$ is the eigenfrequency of the quantum transition,

$$\Omega_e=\frac{\mathbf{DE}}{\hbar},\ \ \Omega_o=2\frac{\mathbf{dE}}{\hbar}\,, \tag{14}$$

$\mathbf{D}=\mathbf{D}_{22}-\mathbf{D}_{11}$ is the permanent dipole moment (PDM), and $\mathbf{d}$ is the transition dipole moment (TDM).

Let us now clarify the physical meaning of dimensionless dynamic variables U, V, and W. The observing dipole moment of atom is determined as $\left\langle\hat{\mathbf{d}}\right\rangle=\mathrm{Tr}\left(\hat{\rho}\hat{\mathbf{d}}\right)$. From this and also from (6) and (7) we obtain $\left\langle\hat{\mathbf{d}}\right\rangle=\bar{\mathbf{D}}+\mathbf{D}W+2\mathbf{d}U$, where $\bar{\mathbf{D}}=\left(\mathbf{D}_{11}+\mathbf{D}_{22}\right)/2$. Therefore, the expression for the polarization response of the medium looks like

$$\mathbf{P}=n\left\langle\hat{\mathbf{d}}\right\rangle=n\left(\bar{\mathbf{D}}+\mathbf{D}W+2\mathbf{d}U\right), \tag{15}$$

where n is the concentration of the two-level atoms.

If the Hamiltonian of an atom has spherical symmetry, then the stationary quantum states are characterized by definite parities. In this case we have $\mathbf{D}=\Omega_{||}=0$ and $\left\langle\hat{\mathbf{d}}\right\rangle=2\mathbf{d}U$. For this reason, the variable U is called the induced dipole moment. Then it follows from (11) that V is the rapidity of the change of dipole moment. In the general case, the contribution to the induced dipole moment is made by the variables U and W.

Multiplying equation (1) scalarly first by $2\mathbf{d}/\hbar$, and then by $\mathbf{D}/\hbar$ and taking into account (14) and (15), we obtain

$$\frac{\partial^2\Omega_o}{\partial z^2}-\frac{n_m^2}{c^2}\frac{\partial^2\Omega_o}{\partial t^2}=\frac{8\pi n}{\hbar c^2}\frac{\partial^2}{\partial t^2}\left(\mathbf{dD}W+2d^2U\right), \tag{16}$$

$$\frac{\partial^2\Omega_e}{\partial z^2}-\frac{n_m^2}{c^2}\frac{\partial^2\Omega_e}{\partial t^2}=\frac{4\pi n}{\hbar c^2}\frac{\partial^2}{\partial t^2}\left(D^2W+2\mathbf{dD}U\right). \tag{17}$$

A further study is based on an approximate analysis of the set of equations (11)–(13), (16), and (17).

We emphasize that the approximation of the electro-dipole interaction between the light field and the atoms imposes certain limitations. It is clear that with the help of this system it is impossible to describe the propagation of X-ray, let alone gamma-radiation, since in these cases the pulsed fields noticeably change on the atomic scales. At the same time, we note that the area of use of system (11)–(13), (16), and (17) is rather wide. This system is able to describe the propagation of pulses in the spectrum that contains frequencies from terahertz to ultraviolet ranges, and their duration lies in the interval from nano- to femtoseconds.

3. MAXWELL-BLOCH EQUATIONS

Let us simplify the system (11)–(13), (16), and (17), assuming the light pulses to be quasimonochromatic at $D = \Omega_e = 0$. If the pulse duration τ_p is such that it contains a large number ($N >> 1$) of light oscillations, it is called quasi-monochromatic. Let T_p be the period of optical oscillations contained in the pulse. Then the quasimonochromaticity condition can also be written by introducing the small parameter:

$$\mu_1 \equiv \left(\omega\tau_p\right)^{-1} << 1. \tag{18}$$

At $\mathbf{D} = \Omega_e = 0$ the system (11)–(13) it is convenient to write in its complex form by introducing the dynamic variable $S \equiv U + iV$:

$$\frac{\partial S}{\partial t} = i\omega_0 S + i\Omega_o W, \quad \frac{\partial W}{\partial t} = \frac{i}{2}\Omega_o\left(S - S^*\right). \tag{19}$$

We represent the pulse field in the form

$$\Omega_o = \psi(z,t)e^{i(\omega t - kz)} + \psi^*(z,t)e^{-i(\omega t - kz)}, \tag{20}$$

where k is the wave number and $\psi(z,t)$ is the complex slowly varying envelope (SVE).

It is easy to see that the introduction of an envelope agrees with the condition (18), which can now be rewritten in the form of inequalities [1]

$$\left|\frac{\partial\psi}{\partial t}\right| << \omega|\psi|, \qquad \left|\frac{\partial\psi}{\partial z}\right| << k|\psi|. \tag{21}$$

Inequalities (21) constitute the essence of the slowly varying envelope approximation (SVEA).

Then we can approximately write

$$\frac{\partial^2\Omega_o}{\partial t^2} \approx \left(2i\omega\frac{\partial\psi}{\partial t} - \omega^2\psi\right)e^{i(\omega t - kz)} + c.c., \tag{22}$$

$$\frac{\partial^2 \Omega_o}{\partial z^2} \approx \left(-2ik \frac{\partial \psi}{\partial t} - k^2 \psi \right) e^{i(\omega t - kz)} + c.c. \tag{23}$$

In the absence of the light pulse field ($\Omega_o = 0$), the first equation (19) describes the motion of a free complex oscillator whose eigenfrequency is ω_0. In this case we have $S \sim \exp\left(i\omega_0 t\right)$. Obviously, this corresponds to the fact that the dipole moment of an excited two-level atom oscillates at the frequency of its quantum transition. The last term on the right-hand side can be regarded as an external force whose frequency is equal to the carrier frequency ω of the light pulse. It is clear that the steady oscillations of the oscillator occur precisely at this frequency. Therefore, we write

$$S(z,t) = R(z,t) e^{i(\omega t - kz)}, \tag{24}$$

where $R(z,t)$ is the complex envelope of dipole moment.

Representing the variable V in the form

$$V = \frac{S - S^*}{2i} = -\frac{i}{2}\left(\mathrm{R}\, e^{i(\omega t - kz)} - \mathrm{R}^*\, e^{-i(\omega t - kz)} \right) \tag{25}$$

and neglecting the derivative of the envelope R, we have

$$\frac{\partial V}{\partial t} = \frac{\omega}{2}\left(R e^{i(\omega t - kz)} + R^* e^{-i(\omega t - kz)} \right).$$

Substituting this expression, as well as (22) and (23) into (16), after equating to each other in the left and right sides of the separate terms containing $e^{i(\omega t - kz)}$ and $e^{-i(\omega t - kz)}$ we obtain

$$\frac{\partial \psi}{\partial z} + \frac{n_m}{c}\frac{\partial \psi}{\partial t} = -i\beta R, \tag{26}$$

where $\beta = \dfrac{4\pi d^2 n \omega_0}{\hbar c n_m}$.

Here we equated the coefficient of the free term ψ to zero, which made it possible to find the dispersion equation $k = n_m \omega / c$.

Thus, the use of the SVEA allowed the wave equation (16) to be reduced from the second order to the first one with respect to the derivatives.

Now we transform the material equations (19). Substituting (24) and (20) into the first equation (19), we have

$$\frac{\partial R}{\partial t} = i\left(\omega_0 - \omega\right) R + i\left(\psi + \psi^* e^{-2i(\omega t - kz)}\right) W .$$

Here the term containing the imaginary exponent that oscillates rapidly on a frequency 2ω can be neglected. This corresponds to the rotating wave approximation (RWA) [1]. Then with good accuracy we write

$$\frac{\partial R}{\partial t} = i\Delta R + i\psi W , \tag{27}$$

where $\Delta = \omega_0 - \omega$ is the frequency detuning of pulse field from a resonant transition.

Substituting (20) and (24) into the second equation (19) and neglecting terms oscillating on the frequency 2ω, we obtain

$$\frac{\partial W}{\partial t} = \frac{i}{2}\left(\psi^* R - \psi R^*\right). \tag{28}$$

The system (26)–(28) is called the Maxwell-Bloch (MB) system. It is integrable and generates solutions in the form of optical solitons [2].

The MB system can be rewritten in terms of real variables, using for the field a representation in the form $\psi = Qe^{i\varphi}$, where Q and φ are real functions having the meaning of the amplitude of the field and its phase, respectively.

We write the envelope of the complex dipole moment in the form $R = \left(u + iv\right)e^{i\varphi}$, where the real variables u and v are called the in-phase and quadrature components of the dipole moment, respectively. Substituting the representations of the envelopes of the field and the dipole moment into (26)–(28), after separating the real and imaginary parts, we obtain

$$\frac{\partial Q}{\partial z} + \frac{n_m}{c}\frac{\partial Q}{\partial t} = \beta v , \quad Q\left(\frac{\partial \varphi}{\partial z} + \frac{n_m}{c}\frac{\partial \varphi}{\partial t}\right) = -\beta u , \tag{29}$$

$$\frac{\partial u}{\partial t} = -\left(\Delta - \frac{\partial \varphi}{\partial t}\right) v , \quad \frac{\partial v}{\partial t} = \left(\Delta - \frac{\partial \varphi}{\partial t}\right) u + QW , \quad \frac{\partial W}{\partial t} = -Qv . \tag{30}$$

The dependence $\varphi(z,t)$ generates in the general case a phase modulation of the optical pulse. Since the complex envelope ψ is slowly varying during the period of oscillations, then $\partial\varphi / \partial t << \omega$. Therefore, the phase modulation of the field can often be neglected. Then the second equation (29) can not be considered at all, but in Eqs. (30) it is necessary to put $\partial\varphi / \partial t = 0$. It is interesting to note that in such form the MB system also turns out to be integrable, generating soliton solutions.

4. RESONANT AND QUASIRESONANT SOLITONS

4.1. SINE-GORDON EQUATION FOR ENVELOPE

Let us consider the case of exact resonance when $\Delta = 0$. We suppose that before the action of the pulse on the medium, all the atoms are in the ground state, *i.e.* $W = -1/2$, when $t = -\infty$. Wherein $U = V = 0$. Consequently, before the impulse action we have $R = u = v = 0$, which corresponds to the absence of an induced dipole moment of the atom. Taking this into account and assuming in (30) that $\Delta = \partial\varphi / \partial t = 0$, we find that $u = 0$ under the action of pulse. Thus, for exact resonance, the in-phase component of the induced dipole moment for the atom is absent. Then the last two material equations (30) take the form $\partial v / \partial t = QW$, $\partial W / \partial t = -Qv$. Introducing the complex function $G = W + iv$, we rewrite them in the form $\partial G / \partial t = iQG$. Then we obtain $G = -(1/2)e^{i\theta}$, where

$$\theta = \int_{-\infty}^{t} Q \, \mathrm{d}t' . \tag{31}$$

After separation in the solution of the real and imaginary parts, we find

$$W = -\frac{1}{2}\cos\theta, \qquad v = -\frac{1}{2}\sin\theta. \tag{32}$$

Substituting the second expression (32) into the first equation (29) and using (31), we obtain the nonlinear sine-Gordon (SG) equation

$$\frac{\partial^2\theta}{\partial z \partial \tau} = -\frac{\beta}{2}\sin\theta . \tag{33}$$

Here we introduced the "local" time $\tau = t - n_m z / c$.

Equation (33) is integrable and has the soliton solutions [2]. The one-soliton solution of equation (33) in the laboratory reference frame has the form

$$Q = \frac{\partial\theta}{\partial t} = \frac{2}{\tau_p}\operatorname{sech}\left(\frac{t - z/\upsilon}{\tau_p}\right), \tag{34}$$

where the soliton velocity is related to its duration by the expression

$$\frac{1}{\upsilon}=\frac{n_m}{c}+\frac{\beta}{2}\tau_p^2. \tag{35}$$

From (34) and (32) we obtain the expression for inversion

$$W=-\frac{1}{2}+\mathrm{sech}^2\left(\frac{t-z/\upsilon}{\tau_p}\right). \tag{36}$$

Thus, the leading edge of the optical pulse transfers the atoms from the ground state to the excited one, and the rear edge induces them back to their original ground state. As a result of the periodic exchange of energy between the light pulse and the medium, an optical envelope soliton is formed. It is clear that time is expended on such a periodic process, therefore, the soliton propagation velocity, determined by the expression (35), is considerably less than the linear velocity c/n_m. This is the essence of the self-induced transparency (SIT) effect, which was discovered experimentally in [3]. In different experiments, velocities υ of two or four orders less than light velocity in vacuum [4] were observed for pico- and nanosecond pulses.

The SIT soliton is also often called the 2π -pulse. The reason for this is that the "area" of a given soliton, defined as $A\equiv\int_{-\infty}^{+\infty}Q\mathrm{d}t$, is equal to 2π. This resonance soliton was the first optical soliton detected experimentally.

By means of the so-called Bäcklund transformations, many-soliton solutions of the SG equation were constructed [5]. The analysis showed that the pulses really elastically interact with each other, restoring their original profiles after collisions.

4.2. HIROTA EQUATION AND NONLINEAR SCHRÖDINGER EQUATION

Let us now consider the case in which the frequency detuning of an optical pulse from a resonance with an ensemble of two-level atoms is different from zero. Suppose that the quasi-resonant inequality [6]

$$\mu_2\equiv\left(\Delta\tau_p\right)^{-1}<<1 \tag{37}$$

is valid.

Using (37), it is possible to exclude material variables from system (27), (28). To do this, we rewrite (27) in the form

$$R=-\frac{\psi}{\Delta}W-\frac{i}{\Delta}\frac{\partial R}{\partial t}. \tag{27a}$$

Since $\partial R/\partial t \sim R/\tau_p$, then the ratio of the second term on the right-hand side of (27a) to its left side is of the order $\sim \mu_2 << 1$. Therefore, we can use the method of successive approximations with respect to the second term on the right-hand side of (27a). In the zeroth approximation we have $R = -\psi W/\Delta$. Substituting this expression into the second term in (27a), we find in the first approximation $R = -\frac{\psi}{\Delta}W + \frac{i}{\Delta^2}\frac{\partial}{\partial t}(\psi W)$. Proceeding as before, we come to the expansion

$$R = -\frac{\psi}{\Delta}W + \frac{i}{\Delta^2}\frac{\partial}{\partial t}(\psi W) + \frac{1}{\Delta^3}\frac{\partial^2}{\partial t^2}(\psi W) - \frac{i}{\Delta^4}\frac{\partial^3}{\partial t^3}(\psi W) + \ldots \quad (38)$$

This expansion is known as the Crisp expansion [7]. Let us now note that the interaction of the pulse with the medium under quasi-resonance conditions is weak. Therefore, in the third and fourth terms on the right-hand side of (33), we neglect the change in the population difference, assuming in these terms that $W = -1/2$. Then, restricting ourselves to the four terms of the expansion, we write

$$R = -\frac{\psi}{\Delta}W + \frac{i}{\Delta^2}\frac{\partial}{\partial t}(\psi W) - \frac{1}{2\Delta^3}\frac{\partial^2 \psi}{\partial t^2} + \frac{i}{2\Delta^4}\frac{\partial^3 \psi}{\partial t^3}. \quad (38a)$$

Since the difference in the populations of the quantum levels of the atom under quasi-resonant conditions varies insignificantly, we substitute (38a) into (28), restricting ourselves to the first three terms of the expansion. Then

$$\frac{\partial W}{\partial t} = \frac{1}{4\Delta^2}\left(\psi^*\frac{\partial \psi}{\partial t} + \psi\frac{\partial \psi^*}{\partial t}\right) - \frac{i}{4\Delta^3}\left(\psi^*\frac{\partial^2 \psi}{\partial t^2} - \psi\frac{\partial^2 \psi^*}{\partial t^2}\right).$$

Integrating this expression we obtain

$$W = -\frac{1}{2}\left[1 - \frac{|\psi|^2}{2\Delta^2} + \frac{i}{2\Delta^3}\left(\psi^*\frac{\partial \psi}{\partial t} - \psi\frac{\partial \psi^*}{\partial t}\right)\right]. \quad (39)$$

Substituting (39) into (38a) and after simple algebraic transformations, we find

$$R = \frac{1}{2\Delta}\left(\psi - \frac{|\psi|^2\psi}{2\Delta^2} + \frac{3i}{2\Delta^3}|\psi|^2\frac{\partial \psi}{\partial t} - \frac{i}{\Delta}\frac{\partial \psi}{\partial t} - \frac{1}{\Delta^2}\frac{\partial^2 \psi}{\partial t^2} + \frac{i}{\Delta^3}\frac{\partial^3 \psi}{\partial t^3}\right). \quad (40)$$

Expression (40) is the expansion of the complex envelope of the induced dipole moment of an atom in powers of nonlinearity and dispersion. The first term inside of brackets (40) determines the linear response. The second term determines the local nonlinear addition to the response, *i.e.* the Kerr nonlinearity, and the coefficient before this term is proportional to the nonlinear susceptibility $\chi^{(3)}$. If $\chi^{(3)} > 0$, then the Kerr nonlinearity is focusing, in the opposite case this nonlinearity is defocusing. As can be seen, in our case the sign of $\chi^{(3)}$ depends on the sign of the detuning Δ. The last two terms characterize different orders of dispersion (see below), the third term corresponds to the dispersion of nonlinearity, and the fourth term determines the linear addition to the group velocity associated with two-level atoms.

Substituting (40) into (26) we will have

$$i\frac{\partial\Phi}{\partial z} = -\frac{k_2}{2}\frac{\partial^2\Phi}{\partial\tau^2} + i\frac{k_3}{6}\frac{\partial^3\Phi}{\partial\tau^3} - a\left|\Phi\right|^2\Phi + ib\left|\Phi\right|^2\frac{\partial\Phi}{\partial\tau}, \tag{41}$$

where

$$\Phi = \psi\exp\left(-i\beta z/2\Delta\right), \tag{42}$$

$\tau = t - z/\upsilon_g$, $k_2 = \beta/\Delta^3$, and $k_3 = 3\beta/\Delta^4$ are the parameters of dispersion of the group velocities (DGV) of second and third degrees, respectively, and the linear group velocity υ_g is determined by the expression $\frac{1}{\upsilon_g} = \frac{n_m}{c} + \frac{\beta}{2\Delta^2}$, $a = \beta/4\Delta^3$, $b = 3\beta/4\Delta^4$.

Equation (36) is known as the Hirota equation [8]. Under the condition $k_2 b = k_3 a$, which is satisfied here, equation (41) turns out to be integrable and has soliton solutions [8].

If we keep on the right-hand side of (41) only the terms of order μ_2^2, *i.e.* we put approximately $k_3 = b = 0$, we obtain the nonlinear Schrödinger (NLS) equation

$$i\frac{\partial\Phi}{\partial z} = -\frac{k_2}{2}\frac{\partial^2\Phi}{\partial\tau^2} - a\left|\Phi\right|^2\Phi. \tag{43}$$

In the general case, the NLS equation describes the propagation of optical solitons in isotropic nonresonant dielectrics having a cubic (Kerr) nonlinearity, which is characterized by the second term on the right-hand side of (43). The model of a two-level medium adopted here is only an illustration of this fact.

Thus, as is carried out in our case, the NLS equation has solutions of the type of “light” solitons (falling to zero at infinity), which in the “laboratory” reference frame have the form

$$\Phi = \frac{1}{\tau_p}\sqrt{\frac{k_2}{a}}\exp\left(i\frac{k_2 z}{2\tau_p^2}\right)\text{sech}\left(\frac{t - z/\upsilon_g}{\tau_p}\right). \tag{44}$$

A distinctive feature of the soliton (44) is that its velocity is not related to its amplitude and duration, but is equal to the linear group velocity corresponding to the carrier frequency. This circumstance makes it possible to use NLS solitons in fiber optical communication lines to transmit information without distortion [9]. Different carrier frequencies of NLS solitons in an optical fiber can correspond to different optical communication channels.

4.3. MODIFIED KORTEWEG – DE VRIES EQUATION AND KONNO – KAMEYAMA – SANUKI EQUATION FOR ENVELOPE

With the shortening of the duration of the optical pulses, it is necessary in (41) to take into account the terms $\sim \mu_2^3$ on the right-hand side and thus depart from the NLS approximation. Therefore, let us now deal in more detail with the Hirota equation (41). Assuming in (41) $\Phi = Qe^{i\varphi}$, after separating the real and imaginary parts from each other, we have

$$\frac{\partial Q}{\partial z} = \frac{k_3}{6}\frac{\partial^3 Q}{\partial \tau^3} + bQ^2\frac{\partial Q}{\partial \tau}, \tag{45}$$

$$Q\frac{\partial \varphi}{\partial z} = \frac{k_2}{2}\frac{\partial^2 Q}{\partial \tau^2} + aQ^3. \tag{46}$$

Equation (45) is called the modified Korteweg-de Vries (MKdV) equation. It turns out to be integrable and has soliton solutions [2]. Its one-soliton solution has the form

$$Q = \frac{1}{\tau_p}\sqrt{\frac{k_3}{b}}\text{sech}\left(\frac{t - z/\upsilon}{\tau_p}\right), \tag{47}$$

where the dependence of the soliton velocity υ on its duration τ_p is given by

$$\frac{1}{\upsilon} = \frac{1}{\upsilon_g} - \frac{k_3}{6\tau_p^2}. \tag{48}$$

From (46) and (47) we have $\varphi = k_2 z / 2\tau_p^2$. Therefore, a soliton solution of the equation (41) looks like

$$\Phi = \frac{1}{\tau_p}\sqrt{\frac{k_3}{b}}\exp\left(\frac{k_2 z}{2\tau_p^2}\right)\operatorname{sech}\left(\frac{t - z/\upsilon}{\tau_p}\right). \tag{49}$$

Let us now consider the case of a two-component medium of two-level atoms interacting with the field of a light pulse. Let one component is in exact resonance with the laser pulse ($\Delta = 0$), and for the other, the quasi-resonance condition (37) is satisfied. Such a situation can arise in a gas mixture of isotopes of a chemical element. Due to the isotopic shift [10], the frequencies of quantum transitions of different isotopes differ somewhat from each other.

It was shown above that in the case of an exact resonance we have $R = ive^{i\varphi}$, and the dynamics of the quadrature component is determined by the expression (32). Then it is easy to see that the presence of a resonant isotopic component leads to the appearance of an additional term $-(\beta/2)\sin\theta$ on the right-hand side of (45), and equation (46) remains unchanged. Bearing in mind that $Q = \partial\theta/\partial\tau$, instead of (45), we have

$$\frac{\partial^2\theta}{\partial z\partial\tau} = -\sigma\sin\theta + b\left(\frac{\partial\theta}{\partial\tau}\right)^2\frac{\partial^2\theta}{\partial\tau^2} + \frac{k_3}{6}\frac{\partial^4\theta}{\partial\tau^4}. \tag{50}$$

Equation (50) is called the Konno-Kameyama-Sanuka (KKS) equation and is integrable if $k_3/b = 4$ [11, 12]. This relation is satisfied in our case (see the expressions for the coefficients of equation (41) immediately after it).

The soliton solution of equation (50) has the form (34). For the soliton velocity, we have

$$\frac{1}{\upsilon} = \frac{1}{\upsilon_g} + \frac{\beta}{2}\tau_p^2 - \frac{k_3}{6\tau_p^2}. \tag{51}$$

Using (39), it is easy to show also that for quasiresonant atoms

$$W = -\frac{1}{2} + \left(\frac{1}{\Delta\tau_p}\right)^2\operatorname{sech}\left(\frac{t - z/\upsilon}{\tau_p}\right). \tag{52}$$

From this and (37) it can be seen that quasiresonant atoms, in contrast to resonance atoms, experience a slight excitation.

It is important to note that this solution is the simplest, one-soliton solution of equation (50). Because of its integrability under the condition $k_3/b = 4$, it also has considerably more complex (for example, many-soliton) solutions [11, 12].

5. REDUCED MAXWELL-BLOCH LIKE SYSTEMS AND RELATED NONLINEAR WAVE EQUATIONS

One of the main trends in the development of laser physics is the creation in the laboratory conditions of light pulses of shorter durations. At present, we can speak of femto- and even attosecond optics [13–22]. There is reason to hope that in the near future the electric fields of such pulses will reach Schwinger values at which electron-positron pairs can be generated from vacuum [21].

A pulse of duration on the order of 1 fs may contain one or several periods of light oscillations. Such signals are called "few-cycle pulses" (FCPs) [14]. In this case, the parameter μ_1 (see (18)) ceases to be small and its value becomes of the order of unity. It is clear that under such conditions it is no longer possible to talk about the envelope of a pulse. Here it is necessary to look for other approaches.

In [23] an alternative to the SVE approach was proposed for describing the phenomenon of SIT. To do this, we used the approximation of a medium of low concentration of two-level atoms, which in our case can be formally represented in the form

$$\mu_3 \equiv \frac{8\pi d^2 n}{\hbar\omega_0} << 1. \tag{53}$$

Taking the typical values $d \sim ea_B \sim 10^{-18}$ CGSE, $\omega_0 \sim 10^{15}\text{s}^{-1}$, we will have from (53) that $n << 10^{23}\text{cm}^{-3}$.

The right-hand sides of (16) and (17) are proportional to the parameter μ_3, and therefore they can be considered small. In such a situation, the unidirectional propagation (UP) method can be applied to the wave equations under consideration, the essence of which is described below. If the right-hand sides of (16) and (17) are set equal to zero, then we have a well-known solution consisting of a superposition of two waves traveling along and opposite the z-axis with velocity c/n_m, respectively. In the approximation (53) the part of the pulse field scattered backward, opposite the *z*-axis, is negligibly small. Therefore, we can assume that the pulse propagates only along the *z*-axis, *i.e.* we have a solution of the form $\Omega_{o,e} = \Omega_{o,e}(t - n_m z/c)$. This assumption allows us to reduce the order of the wave equations (16) and (17). To take into account the right-hand sides of (16) and (17), we introduce the "local" time $\tau = t - n_m z/c$ and the "slow" coordinate. Further we will assume that $\Omega_{o,e} = \Omega_{o,e}(\tau,\zeta)$. Then

$$\frac{\partial}{\partial z} = \frac{\partial\tau}{\partial z}\frac{\partial}{\partial\tau} + \frac{\partial\zeta}{\partial z}\frac{\partial}{\partial\zeta} = -\frac{n_m}{c}\frac{\partial}{\partial\tau} + \mu_3\frac{\partial}{\partial\zeta}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial\tau}.$$

The corresponding second derivatives are obtained by squaring the right-hand sides of these expressions. Neglecting the small term $\sim \mu_3^2$, we write

$$\frac{\partial^2}{\partial z^2} \approx \frac{n_m^2}{c^2}\frac{\partial^2}{\partial \tau^2} - 2\mu_3 \frac{n_m}{c}\frac{\partial^2}{\partial \tau \partial \zeta}, \quad \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial \tau^2}.$$

As a result, after integration, we rewrite (16) and (17) in the form

$$\mu_3 \frac{\partial \Omega_o}{\partial \zeta} = -\frac{4\pi n}{\hbar c n_m}\frac{\partial}{\partial \tau}\left(\mathbf{dD}W + 2d^2U\right),$$

$$\mu_3 \frac{\partial \Omega_e}{\partial \zeta} = -\frac{2\pi n}{\hbar c n_m}\frac{\partial}{\partial \tau}\left(D^2W + 2\mathbf{dD}U\right).$$

Performing a reverse transition to the original independent variables, we obtain

$$\frac{\partial \Omega_o}{\partial z} + \frac{n_m}{c}\frac{\partial \Omega_o}{\partial t} = -\frac{4\pi n}{\hbar c n_m}\frac{\partial}{\partial t}\left(\mathbf{dD}W + 2d^2U\right), \tag{54}$$

$$\frac{\partial \Omega_e}{\partial z} + \frac{n_m}{c}\frac{\partial \Omega_e}{\partial t} = -\frac{2\pi n}{\hbar c n_m}\frac{\partial}{\partial t}\left(D^2W + 2\mathbf{dD}U\right). \tag{55}$$

The procedure for reduction of the wave equations (16), (17) to the form (54), (55) allows us to state that the light pulse can contain an arbitrary number of oscillations. They can be very much, and very little (up to one). In this approximation, the UP advantageously differs from the SVE approximation, which leads to the MB system. On the other hand, from the above procedure of reduction of wave equations from the second order to the first one it follows that the velocity of the optical pulse is slightly different from the linear velocity c / n_m. Therefore, in the UP approximation, it is impossible to describe the significant slowing down of the propagation velocity of a pulse, as is the case with the SIT effect.

5.1. SCALAR RMB SYSTEM

Let the vectors $\mathbf{D}$ and $\mathbf{d}$ are parallel to each other and form the axis of optical anisotropy. Let the light pulse be polarized along this axis and propagate perpendicular to it. Then we can choose any of the equations (54) and (55). Using in these conditions the expression $\Omega_e = \mu\Omega_o$, where $\mu = D / 2d$, and also the system (11)–(13), it is easy to show that the expression in the brackets on the right-hand side of (54) is equal to $-2d^2\omega_0 V$. Then equations (11)–(13) and (54) can be written in the form of a system

$$\frac{\partial U}{\partial t} = -\left(\omega_0 - \mu\Omega_o\right)V , \tag{56}$$

$$\frac{\partial V}{\partial t} = \left(\omega_0 - \mu\Omega_o\right)U + \Omega_o W , \tag{57}$$

$$\frac{\partial W}{\partial t} = -\Omega_o V , \tag{58}$$

$$\frac{\partial \Omega_o}{\partial z} + \frac{n_m}{c}\frac{\partial \Omega_o}{\partial t} = \frac{8\pi d^2 n\omega_0}{\hbar c n_m} V . \tag{59}$$

Hence it is seen that the electric field of the pulse performs two functions: due to the dipole moment of the transition d, it causes a quantum transition between stationary atomic states; due to the permanent dipole moment D, shifts the frequency of this transition.

If $D = \mu = 0$, then the equations (56)–(59) pass into the reduced Maxwell-Bloch (RMB) system. This system is integrable [23], generating soliton and breather solutions. The breather solution differs from the soliton solutions of the traveling wave type discussed above in that the profile of the breather in the accompanying frame is deformed, periodically repeating its shape. The breather solution of the RMB system has the form

$$\Omega_o = \frac{\partial}{\partial t}\left\{ \operatorname{arctg}\left[\frac{1}{\omega\tau_p} \operatorname{sech}\left(\frac{t - z/v_g}{\tau_p} \right) \sin\left(\omega\left(t - z/v_{ph} \right) \right) \right] \right\}, \tag{60}$$

where the nonlinear group v_g and phase v_{ph} velocities are determined by the expressions

$$\frac{1}{v_g} = \frac{n_m}{c} + \beta\tau_p^2 \frac{\left(\omega_0^2 + \omega^2\right)\tau_p^2 + 1}{\left[\left(\omega_0^2 - \omega^2\right)\tau_p^2 + 1\right]^2 + 4\omega^2\tau_p^2} , \tag{61}$$

$$\frac{1}{v_{ph}} = \frac{n_m}{c} + \beta\tau_p^2 \frac{\left(\omega_0^2 - \omega^2\right)\tau_p^2 - 1}{\left[\left(\omega_0^2 - \omega^2\right)\tau_p^2 + 1\right]^2 + 4\omega^2\tau_p^2} . \tag{62}$$

This solution is a two-parameter one, so it contains two free parameters: the duration τ_p of the breather and the central frequency ω of its spectrum. Due to the difference between the group and phase velocities, the profile of the breather is not stationary. As can be seen from (50), the area of this breather equals to zero, *i.e.* $A = \int_{-\infty}^{+\infty} \Omega_o \mathrm{d}t = 0$. In this connection, such a breather is sometimes called an 0π-pulse.

If $\omega\tau_p \sim 1$, then the solution (60)–(62) describes the propagation of a FCP. If $\omega\tau_p >> 1$, then it goes into the envelope soliton of MB system. If the frequencies ω and ω_0 slightly differ from each other, then from (60)–(62) we have in this case

$$\Omega_o = \frac{4}{\tau_p}\mathrm{sech}\left(\frac{t - z/v_g}{\tau_p}\right)\cos\left(\omega\left(t - z/v_{ph}\right)\right),$$

$$\frac{1}{v_g} = \frac{n_m}{c} + \frac{\beta\tau_p^2/2}{1+\left(\Delta\tau_p\right)^2}, \quad \frac{1}{v_{ph}} = \frac{n_m}{c}.$$

With zero detuning ($\Delta = 0$), the expression for the group velocity goes over into the corresponding expression (35) for the soliton of the SG equation.

It was shown in [24] that the system (56)–(59) is also integrable for $\mu \neq 0$. In this form, this system was called “RMB with PDM”. The corresponding soliton solutions are rather cumbersome, so we can not discuss them here. We note only one important detail: the “RMB with PDM” has a solution in the form of a breather with a nonzero area. This result seems to be nontrivial. A solution of the “nonzero” breather type was found numerically in [25]. The corresponding analytical solution was obtained in [26].

5.2. VECTOR SYSTEM “RMB WITH PDM” AND APPROXIMATION OF SUDDEN EXCITATION

Now let the vectors $\mathbf{D}$ and $\mathbf{d}$ be mutually orthogonal. The vector $\mathbf{D}$ is parallel to the axis of optical anisotropy. An extraordinary component Ω_e of the pulse is polarized along the same axis. In turn, the ordinary component is polarized along $\mathbf{d}$, and the *z*-axis is perpendicular to the plane formed by the vectors $\mathbf{D}$ and $\mathbf{d}$. In this case, equations (54) and (55) take the form

$$\frac{\partial\Omega_o}{\partial z}+\frac{n_o}{c}\frac{\partial\Omega_o}{\partial t}=-\frac{8\pi d^2 n}{\hbar c n_m}\frac{\partial U}{\partial t}\,, \tag{63}$$

$$\frac{\partial\Omega_e}{\partial z}+\frac{n_e}{c}\frac{\partial\Omega_e}{\partial t}=-\frac{2\pi D^2 n}{\hbar c n_m}\frac{\partial W}{\partial t}\,. \tag{64}$$

Equations (11)–(13), (63), and (64) form a vector system RMB with PDM.

Here we introduce the ordinary n_o and extraordinary n_e refractive indices, generally speaking, unequal to each other.

As follows from (11)–(13), the ordinary component of the pulse causes a quantum transition, and the extraordinary component shifts the frequency of this transition.

It was shown in [27, 28] that the system of equations (11)–(13), (63), and (64) is integrable under condition $n_e = n_o$, having soliton solutions. Without conducting a detailed analysis here, we show how in the approximation of sudden excitations [29–32].

$$\mu_4 = \omega_0 \tau_p << 1 \tag{65}$$

from this system one can obtain an integrable wave equation, which also has soliton solutions.

It is seen from (11)–(13), (63), and (64) that

$$\left(\frac{\partial}{\partial z}+\frac{n_m}{c}\frac{\partial}{\partial t}\right)\left[\Omega_e^2-2\omega_0\Omega_e+\frac{D^2}{4d^2}\Omega_o^2\right]=0\,, \tag{66}$$

if $n_e = n_o = n_m$. It follows that the expression in square brackets is equal to an arbitrary function of the variable $\tau = t - n_m z / c$. Assuming that $\Omega_{o,e} \to 0$ for $z, t \to \pm\infty$, we come to the conclusion that this function is equal to zero. Then

$$\omega_0 - \Omega_e = \sqrt{\omega_0^2 - \left(\frac{D}{2d}\right)^2 \Omega_o^2}\,. \tag{67}$$

From this it is clear that, in accordance with condition (65), the system (11)–(13) can be rewritten, in the zeroth approximation on the parameter μ_4, in the form

$$\frac{\partial U}{\partial t}=0\,,\;\; \frac{\partial V}{\partial t}=\Omega_o W\,,\;\; \frac{\partial W}{\partial t}=-\Omega_o V\,.$$

Hence we have

$$V = -\frac{1}{2}\sin\vartheta,\ W = -\frac{1}{2}\cos\vartheta, \tag{68}$$

where

$$\vartheta = \int_{-\infty}^{t} \Omega_o \mathrm{d}t'. \tag{69}$$

Here we took into account that before the action of the pulse all the atoms were in the ground state.

Substituting the first expression (68) into the right-hand side of equation (11), we find in the first approximation

$$\frac{\partial U}{\partial t} = \frac{\omega_0 - \Omega_e}{2}\sin\vartheta. \tag{70}$$

Substituting (70) into the right-hand side of equation (63) with allowance for (67) and (69), we obtain the modified sine-Gordon (MSG) equation

$$\frac{\partial^2 \vartheta}{\partial z \partial \tau} = -\beta_o \sqrt{1 - \tau_c^2 \left(\frac{\partial \vartheta}{\partial \tau}\right)^2}\ \sin\vartheta, \tag{71}$$

where $\beta_o = \dfrac{4\pi d^2 n\omega_0}{\hbar c n_m}$, $\tau = t - n_m z / c$,

$$\tau_c = \frac{1}{2\omega_0}\left|\frac{D}{d}\right|. \tag{72}$$

A detailed study of the soliton solution of MSG equation (71) is contained in [33]. Here we only note that the behavior of the solution depends essentially on the relationship between the pulse duration and the characteristic anisotropy time (72).

Assuming $\tau_c << 1$ in (71), we arrive at the sine-Gordon equation obtained for FCP in [30, 31].

5.3. APPROXIMATION OF OPTICAL TRANSPARENCY

In [30, 31, 34] for describing the nonlinear propagation of extremely short pulses it has been proposed the approximation of optical transparency

$$\mu_5 \equiv \left(\omega_0 \tau_p\right)^{-2} << 1. \tag{73}$$

Under such conditions, the interaction of the pulse with the medium is relatively weak. The condition (73) is satisfied by electron-optical transitions, for which $\omega_0 \sim 10^{15} - 10^{16}\text{s}^{-1}$. Then the pulse duration is $\tau_p \sim 10^{-14} s \sim 10$ fs.

Let us first consider the isotropic case when $D = \Omega_e = 0$. Then it is easy to exclude the variable V from (11)–(13):

$$\frac{\partial^2 U}{\partial t^2} + \omega_0^2 U = -\omega_0 \Omega_o W \,, \qquad \frac{\partial W}{\partial t} = \frac{\Omega_o}{\omega_0}\frac{\partial U}{\partial t} \,, \tag{74}$$

According to (73), the first term on the left-hand side of the equation for U is much less than the second one. Therefore, we can apply the method of successive approximations with respect to the first term. Taking into account the weak deviation of W from the equilibrium value $-1/2$, we have in the second order

$$U = -\frac{\Omega_o}{\omega_0} W - \frac{1}{2\omega_0^3}\frac{\partial^2 \Omega_o}{\partial t^2} \,. \tag{75}$$

In order to find the dependence on the pulse field, we substitute (75) into the second equation (74), restricting in (75) only the first term on the right-hand side with the substitution $W \to -1/2$. Then $\frac{\partial W}{\partial t} = \frac{\Omega_o}{2\omega_0^2}\frac{\partial \Omega_o}{\partial t}$. Integrating this equation with allowance for the fact that $W = -1/2$ for $\Omega_o = 0$, we obtain

$$W = -\frac{1}{2}\left(1 - \frac{\Omega_o^2}{2\omega_0^2}\right). \tag{76}$$

Substituting (76) into (75), we arrive at the expression

$$U = \frac{1}{2}\left(\frac{\Omega_o}{\omega_0} - \frac{\Omega_o^3}{2\omega_0^3} - \frac{1}{\omega_0^3}\frac{\partial^2 \Omega_o}{\partial t^2}\right). \tag{77}$$

Here the first term in the brackets on the right-hand side corresponds to the linear contribution to the dipole moment with respect to the field, the second term describes a nonlinear cubic addition to the dipole moment, and the last term determines the temporal dispersion of the atomic response.

After substituting (77) into (63), we have the modified Korteweg-de Vries (MKdV) equation

$$\frac{\partial\Omega_o}{\partial z}-\frac{3}{2}g\Omega_o^2\frac{\partial\Omega_o}{\partial\tau}-g\frac{\partial^3\Omega_o}{\partial\tau^3}=0. \qquad (78)$$

Here $g=\frac{4\pi d^2 n}{\hbar c\omega_0^3}$, $\tau=t-n_0 z/c$, and $n_0=n_m+\frac{4\pi d^2 n}{\hbar\omega_0 n_m}$ is the inertialess refractive index.

Above we have already encountered the MKdV equation (see (45)). However, these two equations contain a fundamental physical difference from each other. Equation (45) is written for the envelope of the pulse field, and (78) for the electric field as a whole.

Equation (78) describes with equal success the propagation of both quasi-monochromatic signals and FCPs. Both these cases are covered by the breather solution of the form (60), where the group and phase velocities are determined by the expressions [2]

$$\frac{1}{v_g}=\frac{n_0}{c}+g\left(3\omega^2-\tau_p^{-2}\right), \qquad \frac{1}{v_{ph}}=\frac{n_0}{c}+g\left(\omega^2-3\tau_p^{-2}\right). \qquad (79)$$

Under condition $\omega\tau_p >> 1$ from the breather solution, we obtain the soliton of the envelope with the carrier frequency ω. If $\omega\tau_p \sim 1$, we have a FCP.

Now, to exclude the material variables from the system (11)–(13), we use the approximation (73) in the anisotropic case, when $D\neq 0$. We will assume that $\Omega_0=0$, *i.e.* we use equation (64) as the wave equation. If we consider the anisotropy to be strong ($D >> d$), we can completely disregard the change and assume that the main contribution to the nonlinearity is made by the PDM of the atom. Then, performing calculations analogous to those carried out in the derivation of equation (78), we obtain the Korteweg-de Vries (KdV) equation

$$\frac{\partial\Omega_e}{\partial z}+q\Omega_e\frac{\partial\Omega_e}{\partial\tau}-g\frac{\partial^3\Omega_e}{\partial\tau^3}=0, \qquad (80)$$

where $q=\frac{4\pi Ddn}{\hbar c n_m\omega_0^2}$, and variables τ and g are defined in the same way as in (78).

Various solutions of equation (80) were studied in detail in [2, 35].

5.4. COMBINATION OF APPROXIMATIONS OF OPTICAL TRANSPARENCY AND SUDDEN EXCITATION

Let us suppose now that an isotropic medium ($D = 0$) consists of two-level atoms of two sorts with concentrations n_1 and n_2. The first type of atoms satisfies the condition (73) and the second type of atoms satisfies the condition (65). Then, assuming the dipole moments of the transitions for both types of atoms to be the same, and combining equations (78) and (71) at $\tau_c = 0$, we obtain the KKS equation [36, 37]

$$\frac{\partial \Omega_o}{\partial z} = g_1 \Omega_o^2 \frac{\partial \Omega_o}{\partial \tau} + g \frac{\partial^3 \Omega_o}{\partial \tau^3} - \beta_o \sin\left(\int_{-\infty}^{\tau} \Omega_o \mathrm{d}\tau' \right), \tag{81}$$

where $g_1 = 3g/2$, and in the expression for g, defined after (78), a replacement $n \to n_1$ should be made, and in the expression for β_o (see (71)) a replacement $n \to n_2$ should be made.

We emphasize that the physical content of equation (81) differs from the physical content of equation (50) in the same way as the physical contents of equations (78) and (45) differ from each other.

Let as assume that $\Omega_o \tau_p << 1$. Then the sine in (81) can be replaced by its argument. In this case we have the equation

$$\frac{\partial \Omega_o}{\partial z} = g_1 \Omega_o^2 \frac{\partial \Omega_o}{\partial \tau} + g \frac{\partial^3 \Omega_o}{\partial \tau^3} - \beta_o \int_{-\infty}^{\tau} \Omega_o \mathrm{d}\tau'. \tag{82}$$

This equation was obtained in [38] using a model of the medium that does not reduce to two-level atoms. For this reason, in [38] $g_1 < 0$ and $g > 0$, *i.e.* the nonlinearity and dispersion created by electron-optical quantum transitions, have, in contrast to the case considered here, different signs.

Suppose now that the characteristic frequency ω of the pulse spectrum satisfies the condition $\omega^4 << \omega_c^4$, where $\omega_c = \left(\beta_o / 3g \right)^{1/4}$ is the characteristic frequency separating the spectral regions of the positive ($\omega > \omega_c$) and negative ($\omega < \omega_c$) DGV [38]. Then the second term on the right-hand side can be neglected in (82) and after differentiation with respect to τ we obtain the Schäfer-Wayne equation [39]

$$\frac{\partial^2 \Omega_o}{\partial z \partial \tau} = \frac{g_1}{3} \frac{\partial^2}{\partial \tau^2} \left(\Omega_o^3 \right) - \beta_o \Omega_o. \tag{83}$$

This equation describes the propagation in transparent dielectrics of FCP whose spectrum belongs to the near infrared range.

As shown in [40], equation (83) is integrable and has solutions of the breather type [39–41].

6. LONG-SHORT WAVE INTERACTION

Let us now consider the case in which one component of the pulse is quasi-monochromatic (envelope pulse) and the other one is broadband (FCP). In particular, this situation describes the optical method of generating terahertz radiation.

It should be noted that at present the topics related to the generation of terahertz radiation are very relevant [42, 43]. This radiation is used in spectroscopy, security systems, image restoration, etc.

We will assume that the ordinary component Ω_o (the short-wave component) of the pulse is quasimonochromatic (see (20)), with an envelope ψ. At the same time, the extraordinary component Ω_e is broadband (the long-wave component). It is easy to see that in this case equation (27) is valid under replacement $\Delta \to \Delta - \Omega_e$. Then, assuming that $\Omega_e << \Delta$, confining ourselves to terms $\sim \left(\Delta \tau_p\right)^{-2}$, in the quasiresonant approximation (37) we arrive at an expansion analogous to (40):

$$R = \frac{1}{2\Delta}\left(\psi - \frac{\Omega_e \psi}{\Delta} - \frac{i}{\Delta}\frac{\partial \psi}{\partial t} - \frac{1}{\Delta^2}\frac{\partial^2 \psi}{\partial t^2}\right). \tag{84}$$

Here we limited ourselves to the second degree of nonlinearity. Expression (39) is simplified to the form

$$W = -\frac{1}{2}\left(1 - \frac{|\psi|^2}{2\Delta^2}\right). \tag{85}$$

Substituting (84) into (63), and (85) into (64), we arrive at the system of Yajima-Oikawa (YaO) equations

$$i\frac{\partial \Phi}{\partial z} = -\frac{k_2}{2}\frac{\partial^2 \Phi}{\partial \tau^2} - b_o \Omega_e \Phi , \tag{86}$$

$$\frac{\partial \Omega_T}{\partial z} = -b_e \frac{\partial}{\partial \tau}\left(|\Phi|^2\right), \tag{87}$$

where $\tau = t - z/\upsilon_g = t - n_e z/c$, $b_o = \dfrac{2\pi d^2 n\omega}{\hbar c n_o \Delta^2}$, $b_e = \dfrac{\pi D^2 n}{2\hbar c n_o \Delta^2}$, the envelope Φ is associated with ψ by means of expression (42), and the expressions for the parameter k_2 and the group velocity υ_g are written immediately after (42).

Here we use the Zakharov-Benney resonance condition [44]

$$v_g = c/n_e, \tag{88}$$

at which the interaction between the two components of the pulse is most effective.

The YaO system turned out to be integrable [45]. Note that the derivation of the given system from (11)–(13), (63), and (64) required the use of all the approximations used above for the derivation of other integrable equations.

The soliton solution of system (86), (87) has the form

$$\Phi = \frac{|k_2|}{\tau_p}\sqrt{\frac{\varepsilon}{b_o b_e}}\exp\left\{i\left[\frac{k_2}{2}\left(\frac{1}{\tau_p^2} - \varepsilon^2\right)z - \varepsilon\tau\right]\right\}\operatorname{sech}\left(\frac{t - z/\upsilon}{\tau_p}\right), \tag{89}$$

$$\Omega_T = -\frac{k_2}{b_o \tau_p^2}\operatorname{sech}^2\left(\frac{t - z/\upsilon}{\tau_p}\right), \tag{90}$$

where the propagation velocity υ of the two-component soliton is given by expression

$$\frac{1}{\upsilon} = \frac{1}{\upsilon_g} - \kappa_2 \varepsilon. \tag{91}$$

The soliton solution (89)–(91) is a two-parameter one. The free parameters are the duration τ_p of the soliton and the coefficient ε characterizing the shift of the carrier frequency of the optical pulse. Since $b_o b_e > 0$, then it follows from (73) that $\varepsilon > 0$. Comparing (89) with (42) and (20), it is easy to see that ε determines the frequency shift into the red spectral area.

Thus, the carrier frequency of the optical pulse decreases after the formation of the optical-terahertz soliton. This can be interpreted as the decay of an optical photon in a nonlinear medium into other optical and terahertz photons. As a result of the frequency shift due to the DGV, the velocity of the optical pulse changes (91). It is seen from (89) that the value of ε is proportional to the square of the amplitude or intensity of the optical pulse. This frequency shift was recorded experimentally [46].

It follows from (87) that if an optical signal is sent as an input to a medium, then it is able to generate a terahertz pulse. Very important for the efficiency of such generation is the fulfillment of condition (88). This condition ensures the generation of terahertz radiation. Deviation from the condition (88) significantly reduces the generation efficiency.

7. CONCLUSION

The various examples considered in this paper indicate an abundance of soliton equations that are used to describe a system made of two-level atoms interacting with an optical field. It is important to note that different situations of such interaction can find non-trivial applications in modern nonlinear optics.

The above excursion into the optical soliton topic revealed that many processes of propagation of pulses of different durations in two-level media are described by integrable equations or coupled systems of equations. In fact, the underlying physics is much more complicated. We have here deliberately considered only one-dimensional cases, when the parameters of the pulse depend only on one spatial variable. In real physical situations, the transverse dimensions of the pulses are finite. The theoretical models that take into account the transverse dynamics of optical pulses are much more complicated than the one-dimensional integrable models. However, the physical models, briefly analyzed in this work, are quite realistic if we notice that the longitudinal dimensions of the solitons considered in this paper are much smaller than the corresponding transverse dimensions. In these cases, the transverse dynamics can be taken into account approximately, starting from one-dimensional soliton solutions. A detailed analysis of such and similar situations is contained in the review [22].

There are good reasons to believe that the simple model of two-level atoms has not yet exhausted itself in the further ability to generate new integrable systems that have solutions in the form of optical solitons.

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