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CAVITY SOLITONS: DISSIPATIVE STRUCTURES IN NONLINEAR PHOTONICS

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Abstract. In the first part of this article, we briefly overview the formation of dissipative structures in various out of equilibrium systems. In the second part, we address the formation of localized structures often called cavity solitons in nonlinear optics. We will focus on the interaction between cavity solitons, the polarization properties, and the effect of delay feedback control. The following systems will be discussed: passive resonators such as optical fibers, whispering gallery mode cavities, integrated ring resonators, left-handed materials, and vertical cavity surface emitting lasers with or without saturable absorption. The year 2017 marks the 50th anniversary of the scientific concept named *dissipative structures* that was put forward by Ilya Prigogine. This paper is dedicated to honor the memory of Ilya Prigogine on the occasion of the anniversary in 2017 of 100 years from his birthday. The aim of this article is to provide a list of selected subjects on some important properties of cavity solitons as generic examples of dissipative structures and their applicability in diverse fields of photonics and nonlinear sciences. Therefore, this paper is meant for active research physicists and engineers working in nonlinear optics and photonics and wishing to have a quick overview of recent developments in terms of the applicability of cavity solitons.

Key words: dissipative structures, cavity solitons, semiconductor lasers, optical resonators.

1. INTRODUCTION

In his talk titled *Structure, Dissipation and Life. Theoretical Physics and Biology*, at Versailles International Conference in 1967, Ilya Prigogine introduced for the first time the concept of *dissipative structures*. The Proceedings of this Conference was published two years later [1]. He demonstrated together with René Lefever that the spontaneous transition from a uniform state to a stationary spatially ordered state requires that the system involves two feedbacks: a positive feedback originating from the auto-catalytical chemical reactions and a negative feedback caused by different diffusion coefficients of the chemical species [2]. They provided a theoretical thermodynamic support of the classical Turing instability [3]. In addition, they showed that the existence of the macroscopic spatial order or dissipative structures is related to a permanent exchange of matter and/or energy with the surrounding [2, 4]. This theoretical support has made the Turing work very popular. Therefore, it is worth to call the instability giving rise to spatially ordered state as Turing-Prigogine instability. The main characteristics of the self-organized structures that emerge from the Turing-Prigogine instability is that their wavelength is intrinsic to the dynamics of the system itself. This means that the wavelength of the emerging dissipative structures is determined only by the dynamical parameters of the system under study and not by the system size or boundary conditions.

Experimental evidence of Turing-Prigogine dissipative structures has been realized thanks to the development of spatial open chemical reactors in the nineties [5–7]. Spatial self-organization has been widely explored in many fields of science such as nonlinear optics, plant ecology, and chemistry. In all these systems, despite of their diversity, the conditions governing the formation of dissipative structures are the same: (i) a non-linear mechanism that generally involves a chemical reaction or light-matter interaction tending to amplify spatial inhomogeneities; (ii) a counteracting phenomenon of transport, such as diffusion, diffraction, dispersion or thermal diffusivity tending to restore spatial uniformity; and (iii) dissipation.

Dissipative structures are not necessary periodic in space. They can be localized in space leading to spatial confinement of the electric field, the biomass density or chemical concentration. They are found in a well-defined region of parameters called a pinning zone [8]. In this regime, the system exhibits a coexistence between two qualitatively different states of behavior: the homogeneous steady states and the spatially periodic dissipative structures. This coexistence occurs thanks to the subcritical nature of the symmetry breaking Turing-Prigogine instability [9–11].

Localized structures (LSs) often called *cavity solitons* in nonlinear photonics may be either isolated, randomly distributed, or self-organized in clusters forming a well-defined spatial pattern. The LSs are homoclinic solutions (solitary or stationary pulses) of partial differential equations. The conditions under which LSs and periodic patterns appear are closely related. This is a universal phenomenon and a well documented issue in various fields of nonlinear science, such as chemistry, plant ecology, and optics, as witnessed by a series of published books and overview papers [12–35]. The emergence of dissipative localized structures in out of equilibrium systems has witnessed tremendous progress in the last years, allowing for the design of photonic devices for all-optical control of light, optical storage, and information processing.

The paper is organized as follows. We discuss in Sec. 2, the formation of localized structures in resonators filled with Kerr media. Temporal localized structures and their application as frequency comb generators will be also addressed in Sec. 2. In Sec 3, we discuss the formation of localized structures in nonlocal media. In Sec. 4 we focus on cavity solitons in semiconductor lasers. We review some recent works on formation of light bullets in optical resonators in Sec. 5. We give our conclusions in Sec. 6.

2. LOCALIZED STRUCTURES AND LOCALIZED PATTERNS IN KERR CAVITIES

In their seminal paper, Luigi Lugiato and René Lefever have derived a mean field model to describe transverse pattern formation in cavity nonlinear optics [36]. The purpose of their article was to analyze spatiotemporal pattern formation in a dissipative, diffractive and nonlinear optical cavity submitted to a continuous laser pump. Early reports on transverse patterns have considered bistable systems subjected to a Gaussian beam [37, 38]. The originality of the Lugiato-Lefever equation, which is now referred to as the LLE, lies in the fact that it predicted the existence of transverse structures of the electric field whose appearance does not require neither a switching process nor an inhomogeneous injected beam. More importantly, Lugiato and Lefever have shown that the mechanism of symmetry breaking leading the spontaneous formation of transverse patterns is fundamentally analogous to the classical Turing-Prigogine instability [3].

The mean field approach used to derive the LLE has permitted to model and to analyze intra-cavity structures in various optical systems using liquid crystals, optical parametric oscillator, second harmonic generation, left-handed materials, photorefractive materials, optical fibers, and phototonic crystal fibers. The field of cavity nonlinear optics has witnessed a constant increase with the explosive growth of the localized structures theme; see, for example, Refs. [12–32, 35].

The LLE applies not only to passive resonators but also to fiber optics [39], whispering gallery mode cavities [40], integrated ring resonators [41, 42], and left-handed materials [43]. In these systems the time and space are considered to be continuous. A discrete version of the LLE to model coupled-waveguide resonators [44, 45], or extended Josephson junction [46] has been also derived.

From the fundamental viewpoint, due to the richness of its dynamical behavior, the LLE has attracted a considerable attention during the last three decades. Two-dimensional optical dissipative structures in the from of hexagonal patterns have been investigated in [47]. The relative stability analysis has shown that rolls and rhombic structures are unstable with respect to perturbations having an hexagonal symmetry [48]. Secondary bifurcations leading to self-pulsating of hexagonal patterns have been predicted in [49, 50]. The influence of noise in the formation of hexagonal patterns has also been studied for that system [51]. Besides periodic patterns, the same mechanism predicts the possible existence of aperiodic, stationary localized structures (LSs). They consist of isolated or randomly distributed peaks in the transverse plane surrounded by regions in the uniform state [10, 52–54]. They have been experimentally observed in intracavity liquid crystals [55] and in fiber Kerr-type media [56]. Self-pulsating LSs in one- and two-dimensional settings have been investigated in [57]. The interaction of self-pulsating LSs that emit weakly decaying dispersive waves, can allow for the formation of bound states [58]. The concept of excitability mediated by LSs has been analyzed in [59]. Polarization properties of periodic [60] and localized structures [61] have been also investigated. The effects of periodic modulation either in space or/and in time of the detuning parameter or the injection have been addressed in [62, 63]. In the large intensity regime, complex spatiotemporal dynamics such as spatiotemporal chaos [64–68] and rogue-wave formation have been realized. In regime devoid of Turing-Prigogine or modulational instability, the switching wave of fronts connecting the homogeneous steady states of LLE has been studied theoretically and/or experimentally in [69–72]. The relative stability of localized structures of multipeak localized patterns [73] as well as the curvature instability affecting the circular shape of LS have been reported [73–75].

On the applied side, the LLE has also been found to be the best framework for the theoretical investigation of Kerr optical frequency comb generation using whispering gallery mode cavities or integrated ring resonators. It should be noted that T. Hänsch and J. Hall [76] have obtained the Nobel Prize in Physics in 2005 for their pioneering work on optical frequency comb technology, and the most recent versions of these combs are theoretically described by the LLE with unprecedented precision (dynamic range higher than 100 dB, and frequency range larger than one octave). These Kerr combs are expected to revolutionize the generation of ultra-stable light beam and microwave signals for aerospace engineering, optical communication networks, and microwave photonic systems.

Optical fibers are considered as privileged media for the study of non-equilibrium instabilities. Indeed, the high light confinement capacity and low loss allow nonlinearities to accumulate during propagation. Energy is supplied from outside of the cavity by injection of a laser beam into a fiber. This simple and at the same time robust device has attracted a lot of attention. The coupling between chromatic dispersion and nonlinearity may be the source of a temporal modulational instability that has been predicted theoretically and confirmed experimentally [77, 78].

Temporal LSs in a driven fiber cavity have been predicted in [10] and observed experimentally [56] to be well suited for bits in an all-optical buffer memory as they can be independently addressed [56]. This work and a recent review paper [79] spurred an interest in alternative semiconductor laser systems with several quite recent experimental demonstrations. In VCSEL systems, Marconi *et al.* [80] realized robust temporal scalar and vector LSs using optical feedback, respectively, from a distant SA mirror and from polarization selective and orthogonal polarization reinjection loops [81]. Gustave *et al.* [82] realized temporal localized structures consisting of 2π phase rotation in a nearly resonant driven highly multimode ring cavity semiconductor laser.

When fiber cavity operates close to the zero wavelength, it is necessary to take into account higher order dispersion. Indeed, several studies indicate that indeed high order dispersion is inherent in the micro-structured fiber often called photonic crystal fiber. High order dispersion fundamentally alters the dynamics of the system by (i) whatever the value of the dispersion of order four, a second frequency associated with the modulation instability appears at the first instability threshold [83] and (ii) the dispersion of order four induces a new modulational instability that allows for the stabilization of large intensity regime [83]. This induces the formation and the stabilization of black localized structures [84–86].

The combination of right-handed and left-handed materials offers the possibility to design devices in which the effective diffraction is zero. Such systems are encountered, for example, in nonlinear optical cavities, where a true zero-diffraction regime could lead to the formation of patterns with arbitrarily small size. In practice, the minimal size is limited by nonlocal terms in the equation of propagation. We have studied the nonlocal properties of light propagation in a nonlinear optical cavity containing a right-handed and a left-handed material. We have derived a model for the propagation, including two sources of nonlocality: the spatial dispersion of the materials in the cavity, and the higher-order terms of the mean field approximation [87, 88]. These results have been applied to a particular case, for which we derived an expression for the parameter fixing the minimal size of the patterns [84]. We have also shown that around the zero-diffraction regime in an optical cavity containing both left- and right-handed materials, one- and two-dimensional dark localized structures can become stable [84]. The stabilization of these structures is the result of high-order diffraction modeled by a bi-Laplacian term with a complex coefficient. We have provided an estimation of this bi-Laplacian term for such a double-layered cavity. In one spatial dimension, the existence of a snaking bifurcation diagram has been demonstrated for these solutions, which shows a larger complexity than generally observed in homoclinic snaking. In two dimensions, numerical simulations have demonstrated a similar coexistence of multiple dips in the intensity profile.

More recently, a new scheme of stabilization, by incorporating the effect of local mode hybridizations, in nonlinear microresonators driven by continuous-wave (cw) lasers has been proposed [89]. This method allows for generation of a stable Turing-Prigogine pattern on a chip-scale with significantly enlarged parameter space, achieving a record-high power-conversion efficiency of 45% and an elevated peak-to-valley contrast of 100. The stationary pattern is discretely tunable across 430 GHz on a THz carrier, with a fractional frequency sideband nonuniformity measured at 7.3×10^{-14}. The simultaneous microwave and optical coherence of the rolls at different evolution stages is demonstrated through ultrafast optical correlation techniques [89]. The on-chip coherent high-power Turing-Prigogine THz system investigated by Huang *et al.* [89] is promising for applications in astrophysics, medical imaging, and wireless communications.

The LLE rapidly arose as a paradigm for understanding periodic dissipative structures, stationary or moving localized patterns, and various complex phenomena in nonlinear optical cavities such as spatiotemporal chaos and rogue-wave formation. The LLE has become an essential model to investigate the dynamical properties of laser fields confined in nonlinear optical resonators. In the last three decades, a considerable progress has been realized in the understanding of several nonlinear phenomena in the framework of the generic LLE.

3. FRONT PROPAGATION IN NONLOCAL MEDIA

Optical nonlinearity is an important property of a material that characterizes its response to an external optical beam. It is generally sufficient to consider that the material response at a certain point in the transverse plane depends only on the value of the light intensity at that point. However, many materials exhibits spatial nonlocality, *i.e.*, the refractive index at a certain point depends not only on the value of the field at that point, but also on the field in the region surrounding this point. The material response if therefore calculated by means of a convolution between the excitation beam and a kernel function accounting for the nonlocality. We classify the kernel functions into two types: weak and strong. If the kernel function decays asymptotically to infinity slower than an exponential function, the nonlocal coupling is said to be strong. Note however, that this definition of strong nonlocal coupling differs from the usual one [90]. Indeed, those authors considered that the nonlocal coupling is strong if the width of the kernel functions is large in comparison with the beam diameter.

One of the phenomena we are interested in, is the motion and the interaction of fronts and more precisely, the impact of strong nonlocal coupling on front propagation in nematic liquid crystals. A front is a heteroclinic connection between equilibrium states, also called walls or wavefronts [91, 92]. Generically, these fronts are moving and interacting, and this type of behavior is called front propagation. When the nonlocal coupling is weak, the interaction of fronts is usually described by the behavior of the tail of one front around the position of the other front. However, for strong nonlocal coupling, the interaction is controlled by the whole influence function and not only by the asymptotic behavior of the front tails.

Recently, we have reported that strong nonlocal interaction is responsible for a new mechanism to stabilize a single localized structure [93]. We showed that strong nonlocal coupling drastically alters the space-time behavior of spatially extended systems by affecting the asymptotic behavior of a single front and by modifying the law governing front interaction. A localized structure that results from a nonlocal coupling possesses a width that significantly increases with the increase of the distance from the Maxwell point. Such plateau-beam localized structures are not limited to optics and photonics. Indeed, they attracted considerable attention in many areas of natural science, such as chemistry, physics, and plant ecology. These stable solutions arise in a dissipative environment and belong to the class of dissipative structures found far from equilibrium. In most cases, the spatial coupling is local for which transport processes like diffraction, dispersion, diffusion, or thermal diffusivity are described by the Laplacian operator. The interplay between these processes and the nonlinearity in dissipative environment, leads to a self-organization phenomenon that is responsible for the formation of either extended or localized patterns. This behavior is not limited to the local coupling, but also occurs in many natural dissipative systems with nonlocal coupling such as firing of cells [94, 95], propagation of infectious diseases [96], chemical reactions [97, 98], population dynamics [99, 100], nonlinear optics [87, 101], granular [102], neural science [103], and plant ecology [104–110]. This issue has been abundantly discussed and is by now fairly well understood. So far however, front dynamics leading to formation of localized structures in a regime far from any pattern forming instability has received only scant attention [111, 112].

Several experimental measurements of a strong nonlocal response in the form of Lorentzian or a generalized Lorentzian have been carried out in nematic liquid crystals cells [113]. The authors have demonstrated that the nonlocal variation of the refractive index cells filled with the E7 liquid crystal is well fitted with a Lorentzian. In this experiment, the thickness of the cell was varied in the range 18–73 microns. Another group of researchers reported on slightly different results using pulsed beams [114]. The nonlocal response of the material was deduced from the interaction of soliton beams in a liquid crystal cell. This response was best fitted with a pseudo-Lorentzian function, which corresponds also to a strong nonlocal coupling, as defined here above. Experimental reconstruction of strong nonlocal coupling has also been performed in photorefractive materials [115]. In this case, the strong nonlinear coupling is originating from the thermal effects.

4. LOCALIZED STRUCTURES IN SEMICONDUCTOR LASERS

In this Section we consider localized structures in semiconductor cavities. These systems have received special attention thanks to the mature semiconductor laser technology and possibility for applications. Owing to the large Fresnel number and the short cavity, Vertical-Cavity Surface-Emitting Lasers (VCSELs) are best suited for potential cavity soliton (CS) studies (for recent reviews see

[25, 27, 31]). VCSELs were first suggested and realized in 1979 [116] and it took more than 10 years to bring them to compatible performance to the edge-emitting lasers [117, 118]. Nowadays, VCSELs are replacing edge-emitting lasers in short and medium distance optical communication links thanks to their inherent advantages: much smaller dimensions, circular beam shape that facilitates coupling to optical fibers, two-dimensional array integration and on wafer testing that brings down the production cost [118]. As VCSELs emit light perpendicular to the surface and the active quantum wells, their cavity length is of the order of the wavelength of laser emission, which imposes single longitudinal mode emission. Thanks to the maturity of the semiconductor technology, VCSELs can be made homogeneous over a size of hundreds of μms while the characteristic CS size is about 10 μm. Furthermore, the timescales of the semiconductor laser dynamics and CS formation are in the ns scale, which allows for fast and accurate gathering of data. However, emission in multiple transverse modes is usually found in small-area VCSELs [119] as a result of spatial hole burning effect [120]. Furthermore, due to the surface emission and cylindrical symmetry VCSELs lack strong polarization anisotropy and may undergo polarization switching between two linearly polarized (LP) fundamental modes [121–124].

There are several ways to create LS in broad-area VCSELs. The most studied one is by injection of a holding beam with appropriate frequency: CSs have been demonstrated in optically injected VCSELs both bellow [125, 126] and above [127, 128] the lasing threshold. Lasing spots spontaneously appear in this system and can be switched on and off by another laser beam. Spatially localized structures have also been found out in medium size VCSELs but only by using their particular polarization properties [129]. Recently, it has been shown that the spin-relaxation mechanism in VCSELs may also lead to observation of vector solitons with slightly elliptical polarization even for LP holding beam [130]. Another approach consists of subjecting the VCSEL to frequency selective optical feedback; in such a system CSs have been demonstrated both experimentally [131] and theoretically [132, 133]. Finally, CSs without holding beam have been also realized by utilizing saturable absorber in face to face coupled VCSELs [134, 135]. In the last two systems, the VCSELs are placed in self imaging optical systems with either an external grating or another VCSEL biased bellow lasing threshold, so that the systems become bistable. As a matter of fact broad area laser with saturable absorber has been the first system in which CSs have been predicted and studied theoretically [136–138]. CSs in a monolithic optically pumped VCSEL with a saturable absorber have been demonstrated in [139] and their switching dynamics studied in [140]. Several applications of CSs in VCSELs have been demonstrated: optical memory [141], optical delay line [142], and optical microscopy [143].

Recently, several theoretical studies appeared on CSs in a VCSEL that is subject to regular, *e.g.* not frequency selective, delayed optical feedback (OF) and capable of supporting CS either by optical injection [144–147] or by incorporated saturable absorber [148–150]. As a matter of fact, temporal dynamics in small-area VCSELs (without spatial effects) has been extensively studied when it has been induced by optical injection [151–157] or by OF [158–160]. The lack of polarization anisotropies [122, 123], and the multitransverse mode behavior of VCSELs provide new features to their steady state and dynamical behavior when they are subject to optical injection and OF, such as polarization switching and mode hopping, regular pulse packages and coherence resonance and polarization chaos [161, 162]. Usually, OF is considered as a single round-trip in the external cavity formed by the VCSEL output mirror and a distant mirror [163]. In a similar way, we have added OF term to the mean field model describing the space-time evolution of broad area VCSEL [164] as a single round-trip term in a self-imaging external cavity (light diffraction while traveling between the VCSEL and the feedback mirror is compensated) [144]. The OF is characterized by the time-delay τ, the feedback strength ζ, and the phase ϕ.

We have demonstrated that OF strongly modifies the input-output characteristic, both by its strength and the phase [144]. The linear stability shows that in contrast to the case without optical feedback where the eigenvalue λ is determined by a third order polynomial, now the eigenvalues are determined by a transcendental equation containing an exponential function, *i.e.*, $\exp(-\lambda\tau)$. In general, that equation does not have analytical solution. Turing-Prigogine instability, which occurs when $\lambda = 0$ for a finite wavenumber $K = K_T$ is also strongly impacted by the OF. When the feedback strength is small, the critical wavenumber K_T at the onset of the Turing-Prigogine instability, which determines the wavelength of the periodic structure $\Lambda_T = 2\pi/K_T$, is almost independent on the feedback phase ϕ however, the threshold of instability is affected. When increasing the OF strength, both the threshold and the wavenumber associated with the Turing-Prigogine instability are strongly affected: depending on the feedback phase ϕ, this threshold can be strongly reduced or enlarged [144]. The feedback phase ϕ dramatically influences the region of modulation instabilities. At a given feedback strength the width of this region can be tuned by slightly changing the external cavity length (in a region of one wavelength) and thus tuning the feedback phase (in a region of 2π). Furthermore, OF may also cause a traveling wave instability, *i.e.* when eigenvalues are a pair of complex conjugate roots with a vanishing real part for a finite wavenumber.

Drift bifurcation in optically injected VCSELs caused by OF has been considered in [144–147]. The dynamical behavior of the spontaneous motion of CS is intimately related to the formation of traveling wave instability. Moving cavity solitons are homoclinic solutions that belong to pinning domain where the uniform branch of stationary homogeneous solution; the lower branch of the input-output characteristics coexists with the traveling wave periodic solutions [165]. In the pinning range of parameters, the lower uniform solution and the traveling wave periodic solution are both linearly stable. As in the case of motionless CSs, in the pinning range of parameters the system exhibits high degree of multistability. We should have a coexistence between not only a moving single peak CS, a uniform solution, and a traveling wave periodic solution, but also a moving multipeak CS. The multiplicity of moving CSs is therefore strongly reminiscent to homoclinic snaking behavior. The profile of a moving CS exhibits a slight asymmetry along the transverse direction. The carrier decay rate γ has a profound influence on the region in the parameter space where drift bifurcation exist; as γ is decreased this region shrinks and disappears. analytical expressions for the threshold and velocity at the drift bifurcation have been derived in [146]. Both the strength and phase of the OF, as well as the carrier decay rate enter in these expressions and explain their profound impact as discussed so far.

Chaotic cavity soliton dynamics in a VCSEL with saturable absorber and time-delayed feedback has been considered in [149, 166, 167]. The mean field model describing the space-time evolution of broad area VCSEL with saturable absorption [148] has been modified to include the delay optical feedback. Fixing all parameters and varying the OF strength the CS is first stationary in time and then exhibits regular time oscillations. The frequency of the oscillations close to the onset of period one dynamics is quite close to the relaxation oscillations of the laser. Increasing further the feedback strength, brings the CS to period two dynamics and finally to chaotic localized dynamics. The transition between different CS profiles is continuous and smooth and the CS does not spread considerably as its peak amplitude changes in time. Despite the complicated chaotic dynamics, the single CS remains firing a single spatially localized structure only. It does not broaden in spite of the available space along the transverse direction. We have checked that the main features of the reported dynamics are robust to noise. More than one CS can be simultaneously present in the system and experience the same kind of dynamics, however not synchronized with each other. It is worth noticing that the bifurcation diagrams of the homogeneous lasing solution of small aperture laser calculated by neglecting diffraction and the single peak CS dynamics resemble each other with the Hopf and the period doubling bifurcations occurring at similar but not exactly the same feedback strengths. Therefore, the CS complex dynamics seems to have its origin in the dynamics of the homogeneous lasing state, *i.e.* the CS are patches of such a state connected to the stable zero (nonlasing) state. This analogy has been further used in [166] and the complete bifurcation analysis has been carried out. Time-delayed feedback induced complex temporal dynamics also takes place in two-dimensional settings.

Dissipative rogue waves in a VCSEL with saturable absorber and time-delayed feedback have been considered in [168, 169]. Recently, spatiotemporal chaos and extreme events have been demonstrated experimentally in an extended microcavity laser in 1D configuration [150]. Here, we consider the control of two-dimensional rogue waves by time-delayed optical feedback. The current is chosen such that the laser without optical feedback resides in a bistable region between the zero homogeneous solution and the lasing solution. For this choice of parameters the upper branch exhibits a subcritical Turing-Prigogine type of bifurcation allowing for the formation of LSs, which experiences a period-doubling bifurcation to spatially localized chaos [149]. Statistical analysis of pulse height distribution of spatial-temporal chaos in the model of a broad-area VCSEL with a saturable absorber reveals a long-tailed statistical contribution that serves as a signature of the presence of rogue waves: rogue waves with pulse heights more than twice the significant wave height appear in the system.

5. LIGHT BULLET FORMATION IN OPTICAL RESONATORS

Light bullets (alias spatiotemporal solitons) [21, 22, 33–35] belong to the class of three-dimensional dissipative localized structures. They have been predicted in Kerr media, taking into account the combined action of diffraction and chromatic dispersion, nonlinearity, pumping, and dissipation. The modeling of this problem leads to a generalization of the Lugiato-Lefever equation [170, 171]. The analytical study of this model has reveled that Kerr media can support periodic solutions in three dimensions (3D) in the form of body-centered cubic lattice structure that are stable over other optical lattices [170, 172, 173]. We have highlighted the existence of localized 3D structures. They consist of a confinement of light in space and in time. These light bullets circulate inside the cavity with the speed of the group velocity of electric fields. This work has been extended to other systems involving optical frequency conversion such as optical parametric oscillator and second harmonic generation process [174, 175]. The light bullets have been also predicted in the wide-aperture laser with saturable absorber [176–178], in systems described by the complex cubic-quintic Ginzburg-Landau equation [179–184], and in a non-mean-field model describing a nonlinear resonator with a saturable absorber [185, 186].

We note that in free propagation, the light bullets are unstable due to beam collapse phenomenon. They are often called light bullets in the case of conservative systems [187, 188] where the stabilization is achieved by a balance among focusing nonlinearity, 2D diffraction, and dispersion. By using a multiscale analysis, Leblond derived a 3D model to describe the propagation of short pulses in quadratic media [189]. The validity conditions of that model including the walk-off, the phase mismatch, and the anisotropy were discussed in [189]. In dissipative systems such as driven nonlinear resonators, a second balance between dissipation and pumping allows for the stabilization of light bullets [170, 171, 174, 175] and is therefore reasonable to name them ”dissipative light bullets” as suggested by Akhmediev in

[190, 191]. There exist other mechanisms for the stabilization of light bullets with respect to collapse phenomenon such as the use of a saturable nonlinearity [191].

Stable three-dimensional solutions in a regime far from any Turing-Prigogine instability have been reported for 3D optical parametric oscillator in [192]. Bifurcation diagrams associated with different types of stable 3D solutions have been constructed, highlighting the phenomenon of multi-stability [192].

The modelling of temporal-spatial dynamics in broad area lasers is very involved and complicated, requiring extensive knowledge of physics, laser dynamics and numerical methods. Diverse theoretical and numerical tools are necessary for the investigation of the formation of light bullets in semiconductors. These include Fourier-domain based and finite-difference time-domain based integration tools to solve two- and three-dimensional partial differential equations, perturbation and different time-scales techniques to simplify the original semiconductor laser equations, and bifurcation theory tools and continuation methods [193].

The experimental observation of light bullets remains a major challenge in nonlinear science. Most of the effort will be concentrated on experimental investigation of light bullets in semiconductors, namely in VCSELs with optical feedback. Quite importantly, VCSELs with a saturable absorber in an external cavity can be mode-locked [194] and can generate temporal solitons [82]. The findings on cavity solitons and temporal solitons in VCSELs open the way for the search for spatiotemporal solitons (light bullets) in these systems.

6. CONCLUSIONS

In this paper, we have highlighted the role of the Turing-Prigogine type of instability in the formation of dissipative structures that can be either periodic or localized in space and/or in time. We have briefly discussed the Turing-Prigogine instability and the theoretical support provided by the Brussels team led by Ilya Prigogine on thermodynamics of far from equilibrium systems. Then we have focused our discussion on the application of dissipative structures in Kerr media such as passive cavities, fiber resonators, whispering gallery mode cavities, and integrated ring resonators. The mean field approach to describe theoretically these devices has potential applications in relation to Kerr optical frequency comb generation.

The field of transverse and temporal effects in nonlinear optics has witnessed an acceleration during the last decades with the explosive growth of the localized structures and solitons themes. In particular, we have focused our overview on cavity solitons in semiconductor lasers. Semiconductor based resonators have the advantage of the mature vertical-cavity surface-emitting lasers (VCSELs) technology for the epitaxial growth of distributed Bragg reflectors and several quantum well or quantum dot layers. This allows wavelength flexibility, output power scalability and mass production. We have discussed the control of cavity solitons in VCSELs by means of time delayed feedback control. We have also presented a review on the formation of dissipative structures in nonlocal media.

Finally, we have reviewed some recent works on formation of light bullets in optical resonators. A challenging direction for further analysis of the three-dimensional dissipative structures is the description of the spatiotemporal localized modes in dissipative systems such as optical resonators. The experimental realization of light bullets will allow for multi-frequency comb generation. This is a research field largely unexploited with many open issues such as the study of domains of stability of spatiotemporal dissipative solitons, their dynamics, interaction, and control.

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