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ROGUE WAVE ON A PERIODIC BACKGROUND FOR KAUP–NEWELL EQUATION

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Abstract. We consider the mixed reduction condition for the Kaup–Newell equation based on the Darboux transformation. By direct calculations, both the first–order rogue wave and the second–order rogue wave on a periodic background are obtained. By taking a certain perturbation coefficient to eigenfunction, we show that the second–order rogue wave can display either a fundamental pattern or a triangle pattern on the periodic background.

Key words: Rogue wave, Kaup–Newell equation, Periodic background.

1. INTRODUCTION

The derivative nonlinear Schrödinger equation

$$q_t + \mathrm{i}q_{xx} + (|q|^2 q)_x = 0, \tag{1}$$

also called the Kaup–Newell (KN) equation [1], has many physical applications, especially in plasma physics and nonlinear optics. It not only governs the evolution of small-amplitude Alfvén waves in a low-β plasma [2–4], but also is used to describe the behavior of large-amplitude magnetohydrodynamic (MHD) waves in a high-β plasma [5, 6]. Besides, it also describes the transmission of sub-picosecond pulses in single mode optical fibers [7, 8].

Besides the KN equation, there are another two derivative-type nonlinear Schrödinger equations. That is, the Chen–Lee–Liu (CLL) equation [9]

$$q_t - \mathrm{i}q_{xx} + |q|^2 q_x = 0, \tag{2}$$

and the Gerdjikov–Ivanov (GI) equation [10]

$$q_t - \mathrm{i}q_{xx} - q^2 q_x^* - \frac{1}{2}\mathrm{i}|q|^4 q = 0. \tag{3}$$

Here the asterisk denotes complex conjugate. Actually, these three equations can be transformed to each other by gauge transformations [11], but the transformations can not preserve the reduction condition in the scattering problem and involve a very complicated integral, which can not be calculated explicitly [12]. Thus, they deserve to be studied separately.

Furthermore, a significant feature for these three equations is that they are integrable. That is, they can be expressed as the compatibility condition of a Lax pair. Based on the Lax pair, the Darboux transformation (DT) and the inverse scattering transformation (IST) can be conducted to these equations. For the KN equation, the IST with zero and nonzero boundary condition was firstly proposed by [1] and [13] respectively, which generated the one–soliton solution, two–soliton solution and paired soliton solution. The further studies of IST with nonzero boundary condition were considered in [14–17], and the N-th order soliton was obtained. The DT for the KN equation was proposed in [18, 19], and the formulas for N-th order solution with nonzero boundary condition of the KN equation were derived, which were expressed as the Vandermonde–like determinants. Based on the Riccati equation and Seahorse function, the N-th breather solutions from non–zero seed were displayed [19]. By applying the Lie algebra splitting and automorphisms, the nonlocal extension of KN equation was given in [20]. The DT of this nonlocal KN equation was obtained in [21], and a kind of global solutions was generated.

Recently, many types of rogue wave solutions, the degenerate cases of breather solutions, were analyzed in a series of works [22–33]. The first–order rogue wave solution of the KN equation was firstly given in [12]. The formulas of the N-th order rogue wave solutions of the KN equation were given in [34–36] with the technique of Taylor expansion. Besides rogue wave solutions, a special rational solution was considered [12], which could generate bright soliton with nonzero boundary, dark soliton with nonzero boundary, and periodic solution. Second–order rational solution, bright soliton with nonzero boundary, were displayed in [34]. These results confirmed the property that was known before that focusing-type equations generate bright solitons, and defocusing-type equations generate dark solitons. However, the mixtures of periodic solutions and rogue-wave solutions were not considered up to now, to the best of our knowledge. Thus, it is interesting and necessary to find explicit solutions describing the behavior of rogue waves on a periodic background.

In general, it is highly nontrivial to construct the rogue waves on a periodic background, which is usually associated with tedious Jacobi elliptic functions [37, 38], integrable equations with variable coefficients [39, 40], PT symmetry [41], or vector integrable equations [42]. These results inspire us to construct the similar solutions by a simple method without using Jacobi elliptic functions or other complicated soliton equations.

In this paper, we will give a direct way to generate rogue waves on a periodic background by applying the odd–th order DT of the KN equation. This paper is organized as follows: In Sec. 2, we review the DT for the KN equation. In Sec. 3, we consider the solutions of the KN equation for the mixed cases. The conclusion and discussion are given in the final Section.

2. THE DT FOR KN EQUATION

The KN equation is the compatibility condition of the following two linear equations, which are called Lax pair equations:

$$\phi_x(x,t,\lambda) = M\phi(x,t,\lambda) = \begin{pmatrix} \mathrm{i}\lambda^2 & q\lambda \\ r\lambda & -\mathrm{i}\lambda^2 \end{pmatrix} \phi(x,t,\lambda), \tag{4}$$

$$\phi_t(x,t,\lambda) = N\phi(x,t,\lambda) = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \phi(x,t,\lambda), \tag{5}$$

where

$$\begin{aligned} A &= 2\mathrm{i}\lambda^4 + \mathrm{i}qr\lambda^2, \\ B &= 2q\lambda^3 + \left(q^2 r - \mathrm{i}q_x\right)\lambda, \\ C &= 2r\lambda^3 + \left(r^2 q + \mathrm{i}r_x\right)\lambda. \end{aligned}$$

Let $r = -q^*$, the compatibility condition of Eqs. (4) and (5) gives the KN equation, *i.e.* $M_t - N_x + MN - NM = 0$ leads to Eq. (1). Moreover, when $r = -q^*$, the Lax pair equations admit symmetry conditions as

- For $\lambda_k = -\lambda_k^*$,

$$f_k^*(x,t,\lambda_k) = g_k(x,t,\lambda_k). \tag{6}$$

- For $\lambda_k = -\lambda_l^*$,

$$f_k^*(x,t,\lambda_k) = g_l(x,t,\lambda_l), \qquad g_k^*(x,t,\lambda_k) = -f_l(x,t,\lambda_l). \tag{7}$$

Here, $\phi_k(x,t,\lambda_k) = \begin{pmatrix} f_k(x,t,\lambda_k) \\ g_k(x,t,\lambda_k) \end{pmatrix}$ $(k = 1,2,\ldots,n)$ are eigenfunctions, solving the Lax pair equation when $\lambda = \lambda_k$.

In general, DT is a special gauge transformation, which preserves the form of the Lax pair. The explicit steps for constructing DT are as following: first, considering the gauge transformation

$$T[1] = T_1\lambda + T_0,$$

where T_1 and T_0 are unknown matrices that are independent of λ. Let $\phi[1] = T[1]\phi$ be a solution of the Lax pair, then

$$T[1]_x + T[1]M = M[1]D[1], \qquad T[1]_t + T[1]N = N[1]D[1]$$

where $M[1]$ and $N[1]$ represent the transformed M and N. By direct calculation, it leads to [12]

$$q[1]=\left(\frac{g_1}{f_1}\right)^2 q+2\mathrm{i}\lambda_1\frac{g_1}{f_1},\qquad r[1]=\left(\frac{f_1}{g_1}\right)^2 r-2\mathrm{i}\lambda_1\frac{f_1}{g_1}. \tag{8}$$

By iteration, the n–order solution can be obtained. Here, we display the formulae directly [12].

Theorem 2.1 *The n-order solution for the KN equation is*

$$q[n]=\frac{\Omega_{11}^2}{\Omega_{21}^2}q+2\mathrm{i}\frac{\Omega_{11}\Omega_{12}}{\Omega_{21}^2}, \tag{9}$$

- *For $n=2k$*

$$\Omega_{11}=\begin{vmatrix}\lambda_1^{n-1}g_1 & \lambda_1^{n-2}f_1 & \lambda_1^{n-3}g_1 & \cdots & \lambda_1 g_1 & f_1\\ \lambda_2^{n-1}g_2 & \lambda_2^{n-2}f_2 & \lambda_2^{n-3}g_2 & \cdots & \lambda_2 g_2 & f_2\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \lambda_n^{n-1}g_n & \lambda_n^{n-2}f_n & \lambda_n^{n-3}g_n & \cdots & \lambda_n g_n & f_n\end{vmatrix},$$

$$\Omega_{12}=\begin{vmatrix}\lambda_1^{n}f_1 & \lambda_1^{n-2}f_1 & \lambda_1^{n-3}g_1 & \cdots & \lambda_1 g_1 & f_1\\ \lambda_2^{n}f_2 & \lambda_2^{n-2}f_2 & \lambda_2^{n-3}g_2 & \cdots & \lambda_2 g_2 & f_2\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \lambda_n^{n}f_n & \lambda_n^{n-2}f_n & \lambda_n^{n-3}g_n & \cdots & \lambda_n g_n & f_n\end{vmatrix},$$

- *For $n=2k+1$*

$$\Omega_{11}=\begin{vmatrix}\lambda_1^{n-1}g_1 & \lambda_1^{n-2}f_1 & \lambda_1^{n-3}g_1 & \cdots & \lambda_1 f_1 & g_1\\ \lambda_2^{n-1}g_2 & \lambda_2^{n-2}f_2 & \lambda_2^{n-3}g_2 & \cdots & \lambda_2 f_2 & g_2\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \lambda_n^{n-1}g_n & \lambda_n^{n-2}f_n & \lambda_n^{n-3}g_n & \cdots & \lambda_n f_n & g_n\end{vmatrix},$$

$$\Omega_{12}=\begin{vmatrix}\lambda_1^{n}f_1 & \lambda_1^{n-2}f_1 & \lambda_1^{n-3}g_1 & \cdots & \lambda_1 f_1 & g_1\\ \lambda_2^{n}f_2 & \lambda_2^{n-2}f_2 & \lambda_2^{n-3}g_2 & \cdots & \lambda_2 f_2 & g_2\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \lambda_n^{n}f_n & \lambda_n^{n-2}f_n & \lambda_n^{n-3}g_n & \cdots & \lambda_n f_n & g_n\end{vmatrix},$$

where Ω_{21} has the same form as Ω_{11}, but f_k, g_k are replaced by g_k, f_k $(k=1,2,3,\cdots,n)$, respectively.

In summary, making the eigenfunctions hold the reduction conditions (6) and (7), the formulae in Theorem 2.1 give the n–order solutions of KN equation.

3. THE SOLUTION OF KN EQUATION OF MIXED CASE

Let $q = -r^*$ and $q = c\exp(\mathrm{i}\rho)$ with $\rho = ax + bt$ and $b = a^2 - ac^2$, solving the Lax pair equation, then it gives

$$\begin{pmatrix} f_k(x,t,\lambda_k) \\ g_k(x,t,\lambda_k) \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\frac{1}{2}\mathrm{i}[\theta_k+\rho]} - \dfrac{2\mathrm{i}\lambda_k^2 - \mathrm{i}a - \mathrm{i}h_k}{2c\lambda_k}\mathrm{e}^{-\frac{1}{2}\mathrm{i}[\theta_k-\rho]} \\ \mathrm{e}^{-\frac{1}{2}\mathrm{i}[\theta_k+\rho]} - \dfrac{2\mathrm{i}\lambda_k^2 - \mathrm{i}a - \mathrm{i}h_k}{2c\lambda_k}\mathrm{e}^{\frac{1}{2}\mathrm{i}[\theta_k-\rho]} \end{pmatrix}, \tag{10}$$

where

$$\theta_k = h_k x + (2\lambda_k^2 + a - c^2)h_k t + 2s_1\epsilon^2, \quad h_k = \sqrt{4c^2\lambda_k^2 + 4\lambda_k^4 - 4a\lambda_k^2 + a^2}, \tag{11}$$

and s_1 denotes the perturbation coefficient.

Based on the eigenfunctions and the formula, the breather solutions and rogue wave solutions can be obtained when n is even. However, in order to consider the mixed case, we need n to be odd.

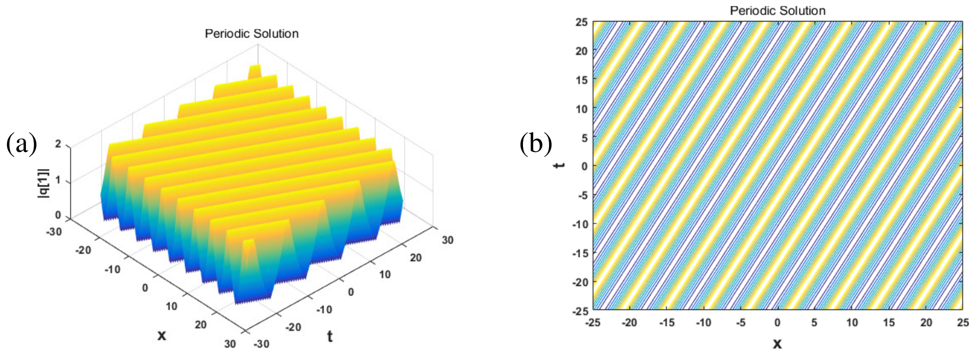

Fig. 1 – (Color online) The dynamics of $q[1]$ (12) with $\beta_1 = 0.5$. (a) The 3-D structure displays a periodic solution. (b) The contour lines of $q[1]$.

Let $n = 1$, $s_1 = 0$ and $\lambda_1 = \mathrm{i}\beta_1$, based on the Theorem 2.1, the first–order solution of KN equation can be obtained as

$$q[1] = \frac{F_1}{G_1^2}\mathrm{e}^{-\mathrm{i}\rho}, \tag{12}$$

with

$$F_1 = (2\sqrt{4\beta_1{}^4+1}+2)\mathrm{e}^{\mathrm{i}\sqrt{4\beta_1{}^4+1}x} + (-8\beta_1{}^4 - 4\beta_1{}^2\sqrt{4\beta_1{}^4+1})\mathrm{e}^{\mathrm{i}\sqrt{4\beta_1{}^4+1}(4\beta_1{}^2t-x)}$$
$$+(-16\beta_1{}^5 - 8\sqrt{4\beta_1{}^4+1}\beta_1{}^3 - 8\beta_1{}^3)\mathrm{e}^{2\mathrm{i}\beta_1{}^2\sqrt{4\beta_1{}^4+1}t},$$
$$G_1 = 2\mathrm{e}^{\mathrm{i}/2\sqrt{4\beta_1{}^4+1}x}\beta_1 + (2\beta_1{}^2 + \sqrt{4\beta_1{}^4+1}+1)\mathrm{e}^{\mathrm{i}/2\sqrt{4\beta_1{}^4+1}(4\beta_1{}^2t-x)}.$$

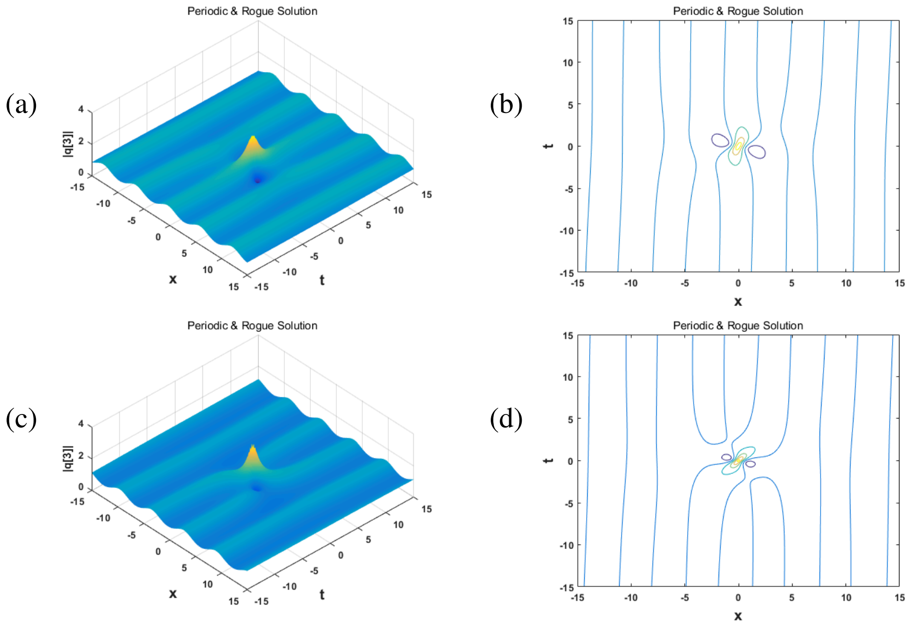

Fig. 2 – (Color online) The dynamics of $q[3]$ displays the mixture of a rogue wave pattern and a periodic pattern with $a=1$ and $c=1$. (a) The rogue wave pattern is located on a single maximum line of the periodic pattern for $\beta=0.1$. (b) The contour lines for $\beta=0.1$. (c) The rogue wave pattern is located between two maxima lines of the periodic pattern for $\beta=-0.1$. (d) The contour lines for $\beta=-0.1$.

Here, we have set $a=1$ and $c=1$ for convenience. From Fig. 1, it is obvious that $q[1]$ is a periodic solution. Actually, $q[1]$ can be a bright soliton and a dark soliton with suitable values of a and c, which has been discussed in detail by Xu [12].

Note that the eigenfunction is degenerated when $\lambda=\frac{1}{2}\sqrt{2a-c^2}-\frac{1}{2}\mathrm{i}c$. In this case, the rogue wave and higher–order rogue wave solutions can be obtained when n is even [12, 34]. In the following, we shall consider the degenerated mixed reductions (6) and (7) for $n=3$. That is, $\lambda_1=-\lambda_2^*=\alpha_1+\mathrm{i}\beta_1=\frac{1}{2}\sqrt{2a-c^2}-\frac{1}{2}\mathrm{i}c$, and $\lambda_3=\mathrm{i}\beta$. By taking Taylor expansion as [12, 34, 35, 43, 44], the third–order solution $q[3]$ is obtained as follows:

$$q[3]=\frac{F_3\,H_3}{{G_3}^2}\exp(-\mathrm{i}\rho), \tag{13}$$

$$\begin{aligned}
G_3 =&G_3^{+}\,\exp\left[\frac{1}{2}\mathrm{i}m^2(2\beta^2t-x)\right]+G_3^{-}\,\exp\left[-\frac{1}{2}\mathrm{i}m^2(2\beta^2t-x)\right],\\
F_3 =&F_3^{+}\,\exp\left[\frac{1}{2}\mathrm{i}m^2(2\beta^2t-x)\right]+F_3^{-}\,\exp\left[-\frac{1}{2}\mathrm{i}m^2(2\beta^2t-x)\right],
\end{aligned}$$

$$
\begin{aligned}
H_3 =& H_3^+ \exp\left[\frac{1}{2}\mathrm{i}m^2(2\beta^2 t - x)\right] + H_3^- \exp\left[-\frac{1}{2}\mathrm{i}m^2(2\beta^2 t - x)\right],\\
G_3^+ =& \left(-4\beta^2 m - 2m^2 - 2m\right)x^2 + \left(-4\mathrm{i}\beta^2 m + 8\mathrm{i}\beta^2 - 2\mathrm{i}m^2 + 2\mathrm{i}m + 4\mathrm{i}\right)x\\
&+ \left(-4\beta^2 m - 2m^2 - 2m\right)t^2 + \left(4\mathrm{i}\beta^2 m + 8\mathrm{i}\beta^2 + 2\mathrm{i}m^2 - 2\mathrm{i}m - 4\mathrm{i}\right)t\\
&- 2\beta^2 m - 8\beta^2 - m^2 - m,\\
G_3^- =& 4\beta m x^2 + (4\mathrm{i}\beta m + 8\mathrm{i}\beta)x + 4\beta m t^2 + \left(16\mathrm{i}\beta^3 + 4\mathrm{i}\beta m\right)t\\
&- 8\beta^3 - 2\beta m - 4\beta,\\
F_3^+ =& -4\beta m x^2 + (4\mathrm{i}\beta m + 8i\beta)x - 4\beta m t^2 + \left(16\mathrm{i}\beta^3 + 4\mathrm{i}\beta m\right)t\\
&+ 8\beta^3 + 2\beta m + 4\beta,\\
F_3^- =& \left(4\beta^2 m + 2m^2 + 2m\right)x^2 + \left(-4\mathrm{i}\beta^2 m + 8\mathrm{i}\beta^2 - 2\mathrm{i}m^2 + 2\mathrm{i}m + 4\mathrm{i}\right)x\\
&+ \left(4\beta^2 m + 2m^2 + 2m\right)t^2 + \left(4\mathrm{i}\beta^2 m + 8\mathrm{i}\beta^2 + 2\mathrm{i}m^2 - 2\mathrm{i}m - 4\mathrm{i}\right)t\\
&+ 2\beta^2 m + 8\beta^2 + m^2 + m,\\
H_3^+ =& \left(-8\beta^3 m - 4\beta m^2\right)x^2 + \left(8\mathrm{i}\beta^3 m + 16\mathrm{i}\beta^3 + 4\mathrm{i}\beta m^2 + 8\mathrm{i}\beta m\right)x\\
&+ \left(-8\beta^3 m - 4\beta m^2\right)t^2 + \left(-8\mathrm{i}\beta^3 m - 4\mathrm{i}\beta m^2 - 8\mathrm{i}\beta\right)t\\
&- 4\beta^3 m + 8\beta^3 - 2\beta m^2 + 4\beta m + 12\beta,\\
H_3^- =& \left(4\beta^2 m - 2m^2 - 2m\right)x^2 + \left(-4\mathrm{i}\beta^2 m + 8i\beta^2 + 2\mathrm{i}m^2 - 2\mathrm{i}m - 4\mathrm{i}\right)x\\
&+ \left(4\beta^2 m - 2m^2 - 2m\right)t^2 + \left(4\mathrm{i}\beta^2 m - 8\mathrm{i}\beta^2 - 2\mathrm{i}m^2 - 6\mathrm{i}m - 4\mathrm{i}\right)t\\
&+ 2\beta^2 m + 16\beta^2 - m^2 + 3m + 4.
\end{aligned}
$$

Here, we have set $a = 1$ $c = 1$ and $m = \sqrt{4\beta^4 + 1}$ for convenience. The pattern of $q[3]$ is displayed in Fig. 2. It shows a rogue wave with periodic background, which is similar to the case [37] for the NLS equation with the Jacobi elliptic function–type seed solution. For $\beta = 0.1$, the rogue wave pattern locates on the area where the periodic pattern reaches to its amplitude. However, for $\beta = -0.1$, the rogue wave pattern locates in the middle of two amplitude trajectories of the periodic pattern, which makes a fake phenomenon that the rogue wave is generated by the interaction of two waves of the periodic pattern.

Similarly, for $n = 5$, assuming $\lambda_1 = \lambda_3 = -\lambda_2^* = -\lambda_4^* = \frac{1}{2}\sqrt{2a - c^2} - \frac{1}{2}\mathrm{i}c$ and $\lambda_5 = \mathrm{i}\beta$, we can get the mixture of second–order rogue wave pattern and the periodic pattern. By considering the perturbation coefficient s_1 (10) as in [34, 35, 44], the second–order rogue wave will be also split into three first–order rogue wave patterns, *i.e.* the triangle structure of second–order rogue wave [35]. We display the patterns of $q[5]$ in Fig. 3 and omit the explicit expression of $q[5]$, because it is very complicated. In Fig. 3, the fundamental structure of second–order rogue wave and triangle structure of second–order rogue wave are clearly observed, and different values of

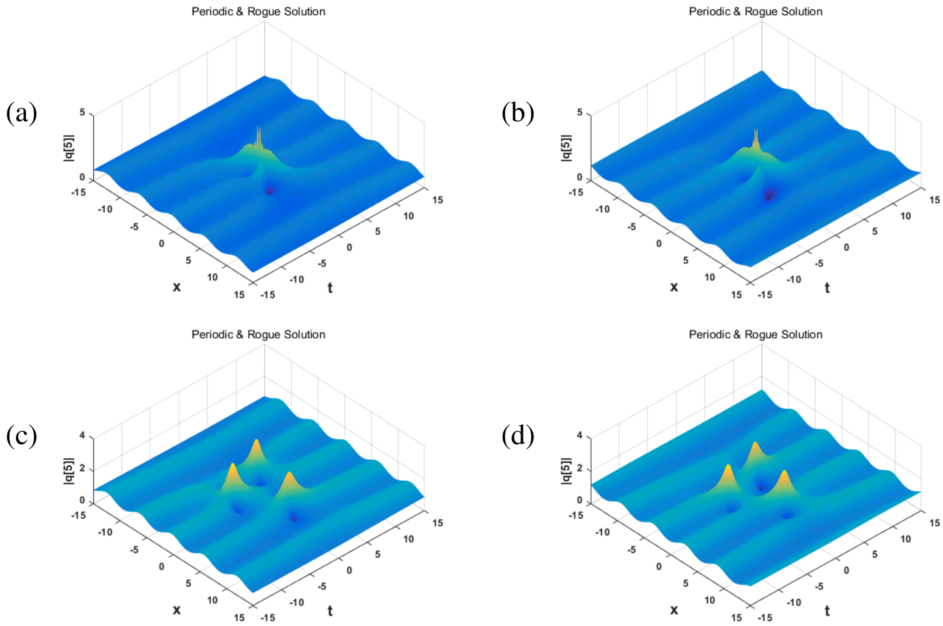

Fig. 3 – (Color online) The dynamics of $q[5]$ displays the mixture of second–order rogue wave pattern and periodic pattern for $a = 1$ and $c = 1$. (a) The fundamental second–order rogue wave pattern and periodic pattern for $\beta = 0.1$. (b) The fundamental second–order rogue wave pattern and periodic pattern for $\beta = -0.1$. (c) The triangle second–order rogue wave pattern and periodic pattern for $\beta = 0.1$, $s_1 = 20$. (d) The triangle second–order rogue wave pattern and periodic pattern for $\beta = -0.1$, $s_1 = 20$.

β generate different mixture of rogue wave and periodic background. Actually, by increasing the value of n, lots of mixed patterns of the KN equation can be obtained by using the above method.

4. CONCLUSIONS

Based on the formulae generated by DT in [12], we consider the odd-th order solution with nonzero boundary condition for the KN equation, and display new kinds of rogue–wave solutions, which locate on a periodic background. These new kinds of rogue waves can display many types of wave patterns as in [35], *i.e.*, the triangle structure, the ring structure, the ring–triangle, and the ring–ring structure. Further, the method used in this paper can be applied to other derivative nonlinear Schrödinger equations including the GI equation and the CLL equation, and we will report the obtained results elsewhere.

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