Romanian Reports in Physics **70**, 108 (2018)

TRAVELING WAVE SOLUTIONS FOR BOUSSINESQ-LIKE EQUATIONS WITH SPATIAL AND SPATIAL-TEMPORAL DISPERSION

MOHAMMAD TAGHI DARVISHI[1,a], MOHAMMAD NAJAFI[1,b], ABDUL-MAJID WAZWAZ[2]

[1]Department of Mathematics, Razi University, Kermanshah 67149, Iran
Email[a]: *mt.darvishi715@gmail.com*
Email[b]: *mnajafi82@gmail.com*
[2]Department of Mathematics, Saint Xavier University, Chicago, IL 60655, USA
Email: *wazwaz@sxu.edu*

Received September 16, 2017

Abstract. This paper obtains traveling wave solutions to four distinct, non-integrable, Boussinesq-like equations with the effect of spatial dispersion for two variants of the Boussinesq equation, and with the effect of spatial-temporal dispersion for other two variants. The sine-cosine method is used to study these equations and to derive the traveling wave solutions.

Key words: Boussinesq-like equation; Traveling solution; Exact solution.

1. INTRODUCTION

The dynamics of shallow water waves along oceans and sea shores are modeled by several forms of nonlinear evolution equations. The phenomenon of shallow-water waves that is seen in various areas like oceans, sea beaches, and lakes is governed by the Boussinesq equation

$$u_{tt} - u_{xx} - (u^2)_{xx} + u_{xxxx} = 0, \tag{1}$$

which describes shallow-water waves moving in both directions [1]. The Boussinesq equation (1) appears in a wide variety of physical systems, such as nonlinear waves in the evolution of perturbations, electromagnetic waves, magneto-sound waves in plasmas, and magneto-elastic waves. Moreover, the Boussinesq equation (1) is integrable by the inverse scattering transformation method, and hence gives multi soliton solutions. This equation also arises in other physical applications such as nonlinear lattice waves, iron sound waves in plasma, and in the study of vibrations of nonlinear strings [1]. Moreover, it was applied to problems in the percolation of water in porous subsurface strata [1]. The Boussinesq equation has been thoroughly examined in the literature. A variety of useful methods was used to investigate this problem. Moreover, the Boussinesq equation was examined numerically and analytically because it possesses a lot of significant properties. Among the techniques that were used are the Hirota's method [2, 3], the modified decomposition method [4, 5], the homotopy perturbation method [5], the extended ansätz function method [6], the homogeneous balance method [7], the sine-cosine ansätz [8], the auxiliary equation method [9], the extended rational expansion method [10], and other methods developed to efficiently solve analytically and numerically both Boussinesq equation and other nonlinear partial differential equations [11–39].

Recently, variants of Boussinesq equation were developed and investigated in the literature; see [1, 8, 27, 30] and some of the references therein. In this work, we will employ the sine-cosine method to study four variants of the Boussinesq equation, namely

$$u_{tt} - u_{xx} - (6u^2u_x + u_{xxx})_x = 0, \tag{2}$$

$$u_{tt} - u_{xx} - (6u^2u_x + u_{xtt})_x = 0, \tag{3}$$

$$u_{tt} - u_{xt} - (6u^2u_x + u_{xxt})_x = 0, \tag{4}$$

and

$$u_{tt} - (6u^2u_x + u_{xxx})_x = 0. \tag{5}$$

It is interesting to note that these four variants are non-integrable equations. In addition, the first and the last equation include spatial dispersion, whereas the second and the third contain spatial-temporal dispersion. The aforementioned variants were given first in [1], and were studied by the Hirota's direct method for single soliton solutions and singular soliton solutions as well. Moreover, other schemes were also used to obtain more solutions with distinct physical structures [1–10].

We aim in this work to apply the sine-cosine method to formally derive traveling wave solutions for the aforementioned Boussinesq-like equations. The parameter constraints that are associated to the new traveling wave solutions will be examined as well.

2. DESCRIPTION OF THE SINE-COSINE METHOD

1. For proper use of the sine-cosine method, we first introduce the wave variable $\xi = x - ct$ into the partial differential equation (PDE)

$$P(u, u_t, u_x, u_{tt}, u_{xx}, u_{tx}, \ldots) = 0, \tag{6}$$

where $u(x,t)$ is a traveling wave solution. This enables us to use the following changes:

$$\frac{\partial}{\partial t} = -c\frac{\partial}{\partial \xi}, \frac{\partial^2}{\partial t^2} = c^2\frac{\partial^2}{\partial \xi^2}, \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}, \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi^2}, \ldots. \tag{7}$$

One can immediately reduce the nonlinear PDE (6) into a nonlinear ordinary differential equation (ODE)

$$Q(u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, \ldots) = 0. \tag{8}$$

The ODE (8) is then integrated as long as all terms contain derivatives, where we neglect integration constants.

2. The solutions of many nonlinear equations can be expressed in the form [40]

$$u(x,t) = \begin{cases} \lambda \sin^\beta(\mu\xi), & |\xi| \le \dfrac{\pi}{\mu}, \\ 0 & \text{otherwise}, \end{cases} \tag{9}$$

or in the form

$$u(x,t) = \begin{cases} \lambda \cos^\beta(\mu\xi), & |\xi| \le \dfrac{\pi}{2\mu}, \\ 0 & \text{otherwise}, \end{cases} \tag{10}$$

where λ, μ, and $m \neq 0$ are parameters that will be determined, and μ and c are the wave number and the wave speed, respectively. We next use the identities

$$\begin{aligned} &u(\xi) = \lambda \sin^m(\mu\xi), \\ &u^n(\xi) = \lambda^n \sin^{nm}(\mu\xi), \\ &(u^n)_\xi = n\mu m \lambda^n \cos(\mu\xi)\sin^{nm-1}(\mu\xi), \\ &(u^n)_{\xi\xi} = -n^2\mu^2 m^2\lambda^n \sin^{nm}(\mu\xi) + n\mu^2\lambda^n m(nm-1)\sin^{nm-2}(\mu\xi), \end{aligned} \tag{11}$$

and, as a result, the derivatives of (10) become

$$\begin{aligned} &u(\xi) = \lambda \cos^m(\mu\xi), \\ &u^n(\xi) = \lambda^n \cos^{nm}(\mu\xi), \\ &(u^n)_\xi = -n\mu m \lambda^n \sin(\mu\xi)\cos^{nm-1}(\mu\xi), \\ &(u^n)_{\xi\xi} = -n^2\mu^2 m^2\lambda^n \cos^{nm}(\mu\xi) + n\mu^2\lambda^n m(nm-1)\cos^{nm-2}(\mu\xi), \end{aligned} \tag{12}$$

and so on for other derivatives.

3. We substitute (11) or (12) into the reduced ODE equation obtained above in (8), balance the terms of the cosine functions when (12) is used, or balance the terms of the sine functions when (11) is used, and solve the resulting system of algebraic equations by using the computerized symbolic calculations. We next collect all terms with same power in $\cos^k(\mu\xi)$ or $\sin^k(\mu\xi)$ and set to zero their coefficients to get a system of algebraic equations among the unknowns μ, m, and λ. Solving the resulting system gives all possible values of the parameters μ, m, and λ [41].

It is interesting to recall the following conclusions [8], where:
(i) For m is a positive fraction, the resulting exact solution is called a *compacton*, defined as soliton with finite wave length or soliton with a compact support free of wave lengths or infinite tails.
(ii) For m is an integer, the resulting exact solution is a *traveling wave solution*.
(iii) For m is a positive fraction, and the resulting solution contains a hyperbolic function, then the solution is called a *solitary pattern solution*.

3. APPLICATIONS

We focus in this Section on applying the sine-cosine method to solve the above mentioned Boussinesq-like equations.

3.1. THE FIRST BOUSSINESQ-LIKE EQUATION

We first study the first Boussinesq-like equation

$$u_{tt} - u_{xx} - (6u^2 u_x + u_{xxx})_x = 0. \tag{13}$$

By applying the traveling wave ansätz $u(x,t) = u(\xi), \xi = kx - ct$ in (13), we obtain

$$(c^2 - k^2)u^{''} - k(6ku^2 u^{'} + k^3 u^{(3)})^{'} = 0. \tag{14}$$

Integrating Eq. (14) twice with respect to ξ and setting the integration constants to zero yields

$$(c^2 - k^2)u - 2k^2 u^3 - k^4 u^{''} = 0, \tag{15}$$

where k and c are constants. Substituting (9) into (15) gives

$$\begin{aligned}\left(c^2 - k^2\right)\lambda \sin^m(\mu\xi) + k^4\mu^2 m^2 \lambda \sin^m(\mu\xi) \\ -\lambda\mu^2 m\,(m-1)\,k^4 \sin^{m-2}(\mu\xi) \\ -2k^2\lambda^3 \sin^{3m}(\mu\xi) = 0.\end{aligned} \tag{16}$$

Equating the exponents and the coefficients of each pair of the sine functions we find the following system of algebraic equations:

$$\begin{aligned}&(m-1) \neq 0, \\ &m - 2 = 3m, \\ &-\lambda\mu^2 m\,(m-1)\,k^4 - 2k^2\lambda^3 = 0, \\ &\left(c^2 - k^2\right)\lambda + k^4\mu^2 m^2 \lambda = 0.\end{aligned} \tag{17}$$

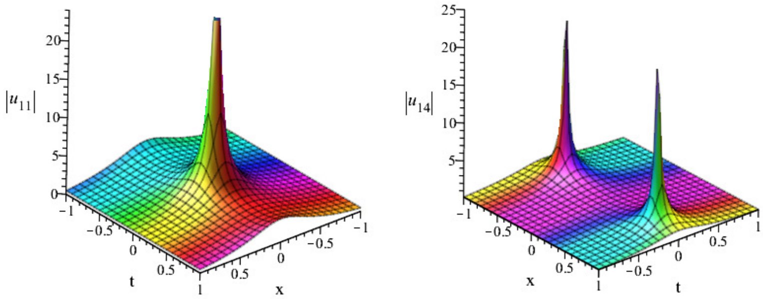

Fig. 1 – Snapshots of traveling wave solutions u_{11} and u_{14} of (13) for $c = 5$ and $k = 2$.

Solving the system (17) yields

$$m = -1 \quad , \quad \lambda = \pm \frac{\sqrt{c^2 - k^2}}{k} \quad , \quad \mu = \pm \frac{\sqrt{k^2 - c^2}}{k^2}. \tag{18}$$

Consequently, the following singular solutions

$$u_{11}(\xi) = \frac{\sqrt{c^2 - k^2}}{k} \csc \left[\frac{\sqrt{k^2 - c^2}}{k^2} (kx - ct) \right], \tag{19}$$

and

$$u_{12}(\xi) = \frac{\sqrt{c^2 - k^2}}{k} \sec \left[\frac{\sqrt{k^2 - c^2}}{k^2} (kx - ct) \right], \tag{20}$$

are readily obtained. It is worth noting that the results (19) and (20) are valid only if $c^2 < k^2$. However, for $c^2 > k^2$, we have the following traveling wave solutions

$$u_{13}(\xi) = \frac{\sqrt{k^2 - c^2}}{k} \text{csch} \left[\frac{\sqrt{c^2 - k^2}}{k^2} (kx - ct) \right], \tag{21}$$

and

$$u_{14}(\xi) = \frac{\sqrt{c^2 - k^2}}{k} \text{sech} \left[\frac{\sqrt{c^2 - k^2}}{k^2} (kx - ct) \right]. \tag{22}$$

As illustrated in Fig. 1, the above solutions represent the singular traveling wave solutions to (13) according to some suitable choices of the parameters.

3.2. THE SECOND BOUSSINESQ-LIKE EQUATION

We next study the second Boussinesq-like equation

$$u_{tt} - u_{xx} - (6u^2 u_x + u_{xtt})_x = 0. \tag{23}$$

Applying the traveling wave ansätz $u(x,t) = u(\xi), \xi = kx - ct$ in (23), yields

$$(c^2 - k^2)\, u'' - k\,(6\,k\,u^2\,u' + k\,c^2\,u^{(3)})' = 0. \tag{24}$$

Integrating Eq. (24) twice with respect to ξ and setting the integration constants to zero yields

$$(c^2 - k^2)\, u - 2\,k^2\,u^3 - k^2\,c^2\,u'' = 0, \tag{25}$$

where k and c are constants. Substituting (9) into (25) gives

$$\begin{aligned} \left(c^2 - k^2\right) \lambda \sin^m(\mu\xi) + k^2\,c^2\mu^2 m^2 \lambda \sin^m(\mu\xi) \\ -\lambda\,\mu^2 m\,(m-1)\,k^2\,c^2 \sin^{m-2}(\mu\xi) \\ -2\,k^2\lambda^3 \sin^{3m}(\mu\xi) = 0. \end{aligned} \tag{26}$$

Equating the exponents and the coefficients of each pair of the sine functions we find the following system of algebraic equations:

$$\begin{aligned} &(m-1) \neq 0, \\ &m - 2 = 3m, \\ &-\lambda\,\mu^2 m\,(m-1)\,k^2\,c^2 - 2\,k^2\lambda^3 = 0, \\ &\left(c^2 - k^2\right)\lambda + k^2\,c^2\mu^2 m^2 \lambda = 0. \end{aligned} \tag{27}$$

Solving the system (27) yields

$$m = -1 \quad , \quad \lambda = \pm\frac{\sqrt{c^2 - k^2}}{k} \quad , \quad \mu = \pm\frac{\sqrt{k^2 - c^2}}{k\,c}. \tag{28}$$

Consequently, the following singular traveling wave solutions

$$u_{21}(\xi) = \frac{\sqrt{c^2 - k^2}}{k} \csc\left[\frac{\sqrt{k^2 - c^2}}{k\,c}\,(k\,x - c\,t)\right], \tag{29}$$

and

$$u_{22}(\xi) = \frac{\sqrt{c^2 - k^2}}{k} \sec\left[\frac{\sqrt{k^2 - c^2}}{k\,c}\,(k\,x - c\,t)\right], \tag{30}$$

are readily obtained. It is worth noting that the results (29) and (30) are valid only if $c^2 < k^2$. However, for $c^2 > k^2$, we have the following solutions

$$u_{23}(\xi) = \frac{\sqrt{k^2 - c^2}}{k} \operatorname{csch}\left[\frac{\sqrt{c^2 - k^2}}{k\,c}\,(k\,x - c\,t)\right], \tag{31}$$

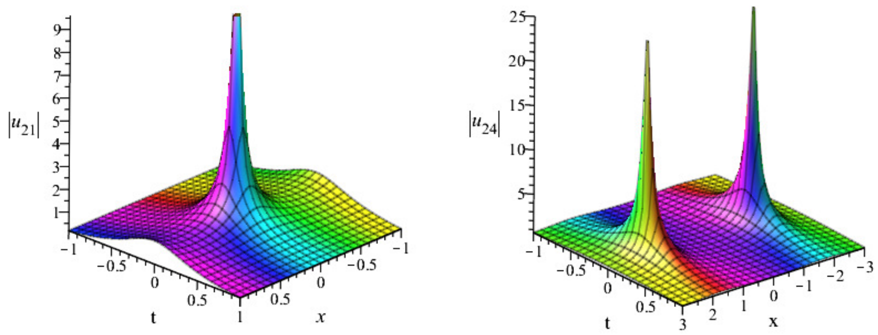

Fig. 2 – Snapshots of traveling wave solutions u_{21} and u_{24} of (23) for $c=2$ and $k=5$.

and

$$u_{24}(\xi)=\frac{\sqrt{c^2-k^2}}{k}\text{sech}\left[\frac{\sqrt{c^2-k^2}}{kc}(kx-ct)\right]. \tag{32}$$

As illustrated in Fig. 2, the above solutions represent singular traveling wave solutions to (23) according to some suitable choices of the parameters.

3.3. THE THIRD BOUSSINESQ-LIKE EQUATION

We next study the third Boussinesq-like equation as follows

$$u_{tt}-u_{xt}-(6u^2u_x+u_{xxt})_x=0. \tag{33}$$

Using the traveling wave ansätz $u(x,t)=u(\xi),\xi=kx-ct$ in (33), gives

$$(c^2+ck)u''-k(6ku^2u'-k^2cu^{(3)})'=0. \tag{34}$$

Integrating Eq. (34) twice with respect to ξ and setting the integration constants to zero yields

$$(c^2+ck)u-2k^2u^3+k^3cu''=0, \tag{35}$$

where k and c are constants. Substituting (9) into (35) gives

$$\begin{aligned}\left(c^2+ck\right)\lambda\sin^m(\mu\xi)-k^3c\mu^2m^2\lambda\sin^m(\mu\xi)\\ +\lambda\mu^2m\,(m-1)\,k^3c\sin^{m-2}(\mu\xi)\\ -2k^2\lambda^3\sin^{3m}(\mu\xi)=0.\end{aligned} \tag{36}$$

Equating the exponents and the coefficients of each pair of the sine functions we find the following system of algebraic equations:

$$\begin{aligned}
&(m-1) \neq 0,\\
&m-2 = 3m,\\
&\lambda \mu^2 m\,(m-1)\,k^3 c - 2k^2\lambda^3 = 0,\\
&\left(c^2+ck\right)\lambda - k^3 c\mu^2 m^2 \lambda = 0.
\end{aligned} \tag{37}$$

Solving the system (37) yields

$$m = -1 \quad , \quad \lambda = \pm\frac{\sqrt{c^2+ck}}{k} \quad , \quad \mu = \pm\frac{\sqrt{k\,(c+k)}}{k^2}. \tag{38}$$

Consequently, for $kc+k^2 > 0$, we have the following solution

$$u_{31}(\xi) = \frac{\sqrt{c^2+ck}}{k}\csc\left[\frac{\sqrt{k\,(c+k)}}{k^2}\,(kx-ct)\right], \tag{39}$$

and

$$u_{32}(\xi) = \frac{\sqrt{c^2+ck}}{k}\sec\left[\frac{\sqrt{k\,(c+k)}}{k^2}\,(kx-ct)\right]. \tag{40}$$

However, for $kc+k^2 < 0$, the following solutions are obtained

$$u_{33}(\xi) = \frac{\sqrt{-c^2-ck}}{k}\operatorname{csch}\left[\frac{\sqrt{-k\,(c+k)}}{k^2}\,(kx-ct)\right], \tag{41}$$

and

$$u_{34}(\xi) = \frac{\sqrt{c^2+ck}}{k}\operatorname{sech}\left[\frac{\sqrt{-k\,(c+k)}}{k^2}\,(kx-ct)\right]. \tag{42}$$

As illustrated in Fig. 3, the above solutions represent traveling wave solutions to (33) according to some suitable choices of the parameters.

3.4. THE FOURTH BOUSSINESQ-LIKE EQUATION

Finally, we consider the fourth Boussinesq-like equation as

$$u_{tt} - (6u^2 u_x + u_{xxx})_x = 0. \tag{43}$$

Applying the traveling wave ansätz $u(x,t) = u(\xi), \xi = kx-ct$ in (43), yields

$$c^2 u'' - k\,(6ku^2u' + k^3 u^{(3)})' = 0. \tag{44}$$

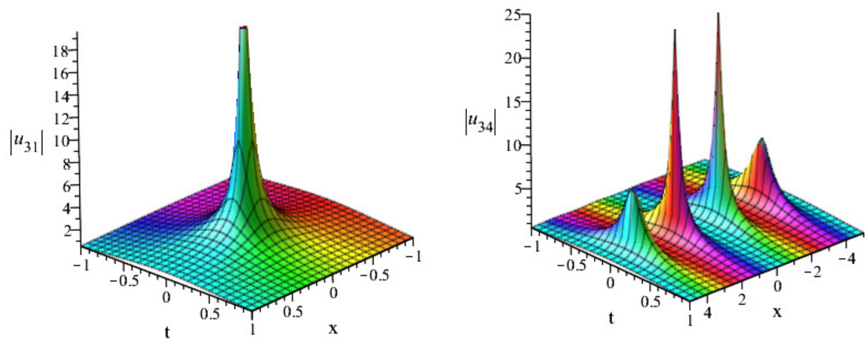

Fig. 3 – Snapshots of traveling wave solutions u_{31} and u_{34} of (33) for $c = 2$ and $k = 3$.

Integrating Eq. (44) twice with respect to ξ and setting the integration constants to zero gives

$$c^2 u - 2k^2 u^3 - k^4 u'' = 0, \tag{45}$$

where k and c are constants. Substituting (9) into (45) gives

$$\begin{aligned} c^2\lambda \sin^m(\mu\xi) + k^4 \mu^2 m^2 \lambda \sin^m(\mu\xi) \\ -\lambda\mu^2 m\,(m-1)\,k^4 \sin^{m-2}(\mu\xi) \\ -2k^2\lambda^3 \sin^{3m}(\mu\xi) = 0. \end{aligned} \tag{46}$$

Equating the exponents and the coefficients of each pair of the sine functions we find the following system of algebraic equations:

$$\begin{aligned} &(m-1) \neq 0, \\ &m - 2 = 3m, \\ &-\lambda\mu^2 m\,(m-1)\,k^4 - 2k^2\lambda^3 = 0, \\ &c^2\lambda + k^4\mu^2 m^2 \lambda = 0. \end{aligned} \tag{47}$$

Solving the system (47) yields

$$m = -1 \quad , \quad \lambda = \pm\frac{c}{k} \quad , \quad \mu = \pm\frac{ci}{k^2}. \tag{48}$$

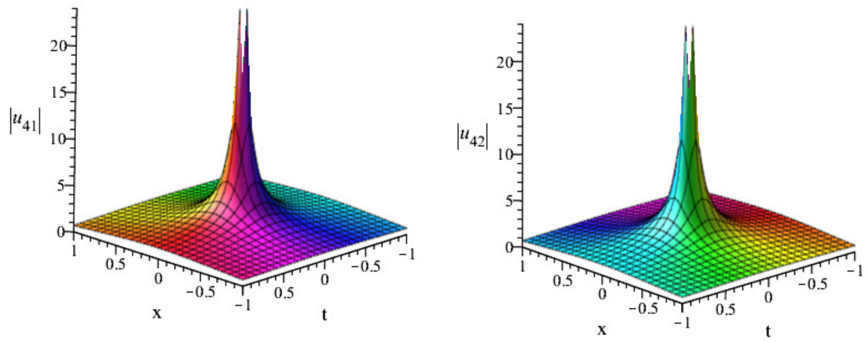

Fig. 4 – Snapshots of traveling wave solutions u_{41} and u_{42} of (43) for $c = 1$ and $k = 1$.

Consequently, we have the following solutions

$$u_{41}(\xi) = \frac{ci}{k}\text{csch}\left[\frac{c}{k^2}(kx - ct)\right], \tag{49}$$

and

$$u_{42}(\xi) = -\frac{ci}{k}\text{csch}\left[\frac{c}{k^2}(kx - ct)\right]. \tag{50}$$

As illustrated in Fig. 4, the above solutions represent traveling wave solutions to (43) according to some suitable choices of the parameters.

4. CONCLUSIONS

In this paper, we employed the sine-cosine method to obtain traveling wave solutions for four distinct Boussinesq-like equations. The obtained results include singular traveling wave solutions, which involve trigonometric functions and hyperbolic functions as well. We examined all constraints that guarantee the existence of these new exact solutions. The obtained results are going to be very useful in various areas of physics and applied mathematics such as fluid dynamics, nonlinear optics, plasma physics and others. To the best of our knowledge, the results obtained in this paper have not been reported in other studies on the Boussinesq-like equations.

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