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SIMILARITY SOLUTIONS OF FIELD EQUATIONS WITH AN ELECTROMAGNETIC STRESS TENSOR AS SOURCE

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Abstract. We study the general relativity with an electromagnetic stress tensor, as source, and Maxwell's equations in curved space. We investigate the similarity solutions of field equations of these two areas by using the generalized symmetry method based on Fréchet derivative of the differential operators. Metrics and electromagnetic fields as functions of two independent variables are considered. The field equations are presented in a scientific form and certain exact solutions of these equations are systematically derived. The results are achieved by obtaining the infinitesimals of the group of transformations, which leave the system of field equations invariant. An optimal system of conjugacy inequivalent subgroups is then identified with the adjoint action of the symmetry group. This is further used to reduce the system of field equations into a system of ordinary differential equations with the aim of deriving certain exact solutions.

Key words: General Relativity, Field Equations, Similarity Solutions, Symmetry Analysis.

1. INTRODUCTION

The general relativity is a unified theory of space, time, and gravitation. The theory's roots extend over almost the entire previous history of physics and mathematics. The laws of general relativity, Einstein's equations, when written in any system of local coordinates, constitute, a non-linear system of partial differential equations for the metric components. In practice, one of the great difficulties of relating the particular features of general relativity to real physical problems, arises from the high degree of non-linearity of Einstein field equations. Therefore, it becomes very difficult to solve these equations unless certain symmetry restrictions are imposed on some space-time metric. These symmetry restrictions are expressed in terms of isometries possessed by space times. These isometries, which are also called Killing vectors, give rise to conservational laws. The symmetries in general relativity have been the subject of several studies in recent years, partly because of the considerable simplification of Einstein's equations resulting from the assumption of one or more symmetries, partly because of interest in the geometric significance of the symmetries, which are described by vector fields of certain geometrical objects on the manifold, and partly because of the possible physical significance of the existence of these symmetries.

Here, we will work with the field equations of general relativity plus electromagnetism

$$R_{ij} - \frac{1}{2} g_{ij} R = k T_{ij}, \tag{1}$$

with

$$k = 8\pi G c^{-4}, \tag{2}$$

and

$$T_{ij} = (4\pi)^{-1} (F_{il} F_j^l - \frac{1}{4} F_{lm} F^{lm} g_{ij}), \tag{3}$$

where F_{ij} is the electromagnetic tensor, g_{ij} is the metric tensor and conventions as to the metric signature [1]. The Einstein-Maxwell equations [2] in curved space are

$$\begin{aligned} &[ijkl] \tfrac{\partial F_{jk}}{\partial x^l} = 0, \\ &\tfrac{\partial}{\partial x^k} [(-g)^{\frac{1}{2}} g^{ij} g^{kl} F_{jl}] = 0 \end{aligned} \tag{4}$$

where $[ijkl] = (+1, -1)$ for (even, odd) permutation of $i, j, k, l = 0$ if any two of i, j, k, l are equal.
We now assume the metric to be diagonal

$$g_{ij} = \delta_{ij} e_i \exp(2 f_i), \tag{5}$$

with

$$e_0 = -1, \;\; e_1 = e_2 = e_3 = 1. \tag{6}$$

It should be noted that there are alternative ways of defining potentials. We then set

$$G_{ij} = A_{i,j} - A_{j,i}, \quad (i, j = 0, 1, 2), \tag{7}$$

and

$$G_{i3} = e_i \exp(f_i - f_j - f_k + f_3)(B_{j,k} - B_{k,j}), \tag{8}$$

$(i, j, k = 0, 1, 2)$ in cyclic order. Here, we introduce a new potential C as $A = C\cos\alpha$ and $B = C\sin\alpha$. The metric [3] is given as

$$-ds^2 = V^{-2}(\exp(2\xi))((-dx^0)^2 + (dx^1)^2) + (x^1)^2 V^{-2} (dx^2)^2 + V^2 (dx^3)^2. \tag{9}$$

With the above substitutions [3], the Einstein-Maxwell field equations become

$$V_{11} + \frac{V_1}{x^1} - V_{00} = \frac{1}{V}(V_1^2 - V_0^2 + C_0^2 - C_1^2), \tag{10}$$

$$C_{11} + \frac{C_1}{x^1} - C_{00} = \frac{2}{V}(V_1 C_1 - V_0 C_0), \tag{11}$$

$$\xi_0 = 2x^1 V^{-2}(V_0 V_1 - C_0 C_1), \tag{12}$$

$$\xi_1 = x^1 V^{-2}(V_0^2 + V_1^2 + C_0^2 + C_1^2), \tag{13}$$

$$\xi_{11} - \xi_{00} = V^{-2}(V_0^2 - V_1^2 + C_0^2 - C_1^2), \tag{14}$$

where the lower subscripts 1 and 0 denote partial differentiation with respect to the corresponding variable x^1, x^0. Equation (14) is derivable from Eqs. (10)-(13); ξ_{01} as calculated from Eq. (12) is identical with that calculated from Eq. (13), with Eqs. (10) and (11) assumed. Thus we can find ξ from V and C, and Eqs. (10) and (11) provide the main equations. These equations can be clearly viewed to be quasilinear wave equations for V and C. The nonlinear system of partial differential equations (PDEs) (10)-(14) describes mathematically and physically important phenomena for electromagnetic fields and gravitational fields in the theory of general relativity. Hence symmetries and exact solutions of this nonlinear system are of great importance.

The literature abounds with many different techniques that have been invoked in an effort to obtain new exact solutions of the Einstein field equations [4–7]. There is a considerable number of exact solutions of the Einstein-Maxwell equations in the literature, for more details we refer to [8]. The powerful methods for finding symmetries and constructing exact solutions, which were used so far, include the Lie classical approach [9, 10], the symmetry reduction approach [11, 12], isovector field method [13, 14], and nonclassical approach [15]. Also, it is worth to mention that recently many of the researchers, working in the area of nonlinear PDEs, have developed many powerful methods, which are further utilized for exploring solutions of various PDEs [16–23] in pure and applied sciences and engineering. The paper is structured as follows: In Sec. 2, the generalized symmetry method is used to derive the Lie symmetries of nonlinear equations (10) and (11). The optimal system of non-conjugate sub-algebras of the full symmetry algebra is identified under the adjoint action of the symmetry group in Sec. 3. Section 4 is devoted to finding the reduced system of ordinary differential equations (ODEs) using various Lie ansätze associated with each basic field in the optimal system of sub-algebras. The systems of reduced ODEs are examined for certain exact solutions. Finally, in the last Section we make some concluding remarks.

2. THE SYMMETRY GROUP AND OPTIMAL SYSTEM

In order to determine the Lie group of transformations of Eqs. (10) and (11), we exploit the generalized symmetry method due to Steinberg [11], which is based on the Fréchet derivatives of the nonlinear operators.

Let the Eqs. (10) and (11) be considered as a manifold $\bar{M} = M_1,\ M_2$,

$$\begin{aligned} M_1(V,C) &= V_{11} + \tfrac{V_1}{x^1} - V_{00} - \tfrac{1}{V}(V_1^2 - V_0^2 + C_0^2 - C_1^2) = 0, \\ M_2(V,C) &= C_{11} + \tfrac{C_1}{x^1} - C_{00} - \tfrac{2}{V}(V_1C_1 - V_0C_0) = 0, \end{aligned} \tag{15}$$

in the space of variables $\bar{X} = (x^1, x^0),\ \bar{\eta} = (V,C)$.

The one-parameter group of local point transformations that leaves Eqs. (10) and (11) invariant corresponds to the vector fields of the form

$$W = A(\bar{X},\bar{\eta})\frac{\partial}{\partial x^1} + B(\bar{X},\bar{\eta})\frac{\partial}{\partial x^0} - D(\bar{X},\bar{\eta})\frac{\partial}{\partial V} - E(\bar{X},\bar{\eta})\frac{\partial}{\partial C}. \tag{16}$$

The group infinitesimals A, B, D, and E are to be found under the following conditions:

$$F_i(\bar{M}_i, \bar{\eta}, \bar{S})|_{\bar{N}=0} = \bar{0}, \tag{17}$$

for $i = 1,\ 2$.

In Eq. (17), $F_i(\bar{M}_i, \bar{\eta}, \bar{S})$ denotes the Fréchet derivative of M_i at $\bar{\eta} = (V,C)$ in the direction of the quasi-linear symmetry operator $\bar{S} = (S_1,\ S_2)$ and is defined by

$$\bar{F}(\bar{N}, \bar{\eta}, \bar{S}) = \frac{d[\bar{N}(\bar{\eta} + \epsilon\bar{\eta}_1)]}{d\epsilon}|\epsilon = 0. \tag{18}$$

The symmetry operator $\bar{S} = (S_1,\ S_2)$ has the following form:

$$S_1(V) = A(\bar{X},\bar{\eta})\frac{\partial V}{\partial x^0} + B(\bar{X},\bar{\eta})\frac{\partial V}{\partial x^1} + D(\bar{X},\bar{\eta}), \tag{19}$$

$$S_2(V) = A(\bar{X},\bar{\eta})\frac{\partial C}{\partial x^0} + B(\bar{X},\bar{\eta})\frac{\partial C}{\partial x^1} + E(\bar{X},\bar{\eta}), \tag{20}$$

Equation (18) is used to find the Fréchet derivative of each of the three nonlinear operators defined through Eq. (15), and we arrive at the following expressions:

$$\begin{aligned} F_1(\bar{M}_1, \bar{\eta}, \bar{S}) = {} & S_1(V_{11} + \tfrac{V_1}{x^1} - V_{00}) + V([S_1]_{11} + \tfrac{[S_1]_1}{x_1} - [S_1]_{00}) \\ & -2(V_1[S_1]_1 - V_0[S_1]_0 + C_1[S_2]_1 - C_0[S_2]_0), \end{aligned} \tag{21}$$

$$\begin{aligned} F_2(\bar{M}_2, \bar{\eta}, \bar{S}) = {} & S_1(C_{11} + \tfrac{C_1}{x^1} - C_{00}) + V([S_2]_{11} + \tfrac{[S_2]_1}{x_1} - [S_2]_{00}) \\ & -2(V_1[S_2]_1 + [S_1]_1C_1 - V_0[S_2]_0 + [S_1]_0C_0). \end{aligned} \tag{22}$$

In view of conditions (17), Eqs. (21) and (22) are used to get the determining equations for the group infinitesimals A, B, D, and E. In other words, Eqs. (21) and (22) are expanded and the temporal derivatives of $V(x^1, x^0)$ and $C(x^1, x^0)$ are substituted with the help of Eqs. (10) and (11). This leads to the polynomial expressions in various partial derivatives of $V(x^1, x^0)$ and $C(x^1, x^0)$ with respect to the spatial variable. On equating the coefficients of various derivative terms to zero in these expressions, a set of determining equations for the group infinitesimals A, B, D, and

E are obtained. Without going into the details of algebraic calculations, we list here the simplified version of the determining equations. The set of equations obtained from Eq. (21) is as follows:

$$\begin{aligned}
&A_V = 0,\ \ A_C = 0,\ \ B_V = 0,\ \ B_C = 0,\\
&A_1 - B_0 = 0,\ \ A_0 - B_1 = 0,\\
&VD_{0C} + E_C = 0,\ \ VD_{VC} + E_V = 0,\\
&D + V^2 D_{VV} - VD_V = 0,\ \ 2D_{1C} + E_1 = 0,\\
&-D - VD_V + V^2 D_{CC} + 2VE_C = 0,\ \ x^1 D_{11} - x^1 D_{00} + D_1 = 0,\\
&Vx^1 A_{11} + 2x^1 D_0 - Vx^1 A_{00} - 2Vx^1 D_{0V} + VA_1 = 0,\\
&-2(x^1)^2 D_1 + 2V(x^1)^2 D_{1V} - Vx^1 B_1 + VB + V(x^1)^2 B_{11} - VB_{00} = 0.
\end{aligned} \tag{23}$$

Similarly, equation (22) brings-in the following additional equations. It is being mentioned here that these equations have been obtained keeping in view the consequences on the infinitesimals as affected by the set of equations (23):

$$\begin{aligned}
&VE_{1V} - E_1 = 0,\ \ -VD_{0V} + E_0 = 0,\\
&VE_{VV} - E_V = 0,\ \ x^1 E_{11} + E_1 - x^1 E_{00} = 0,\\
&-VD_V + V^2 E_{VC} + D = 0,\ \ VE_{CC} - E_V - 2D_C = 0,\\
&2V(x^1)^2 E_{1C} - V(x^1)^2 B_{00} - 2(x^1)^2 D_1 - Vx^1 B_1 + VB + V(x^1)^2 B_{11} = 0,\\
&-D - 2VB_1 - V^2 E_{VC} + VD_V + 2VA_0 - V^2 A_{0V} = 0,\\
&-2Vx^1 E_{0C} - Vx^1 A_{00} + Vx^1 A_{11} + 2x^1 D_0 + VA_1 = 0.
\end{aligned} \tag{24}$$

Now, the two sets of equations (23) and (24) are combined, and simplified to the extent possible for the determination of the infinitesimals A, B, D, and E. Without presenting any calculations, we provide the following form of the generalized symmetries:

$$\begin{aligned}
&A = a_1 x^0 + a_2,\\
&B = a_1 x^1,\\
&D = -2(a_3 C - \tfrac{a_4}{2})V,\\
&E = (-C^2 + V^2)a_3 + a_4 C + a_5,
\end{aligned} \tag{25}$$

where $a_j, j = 1, 2, 3....5$ are arbitrary constants. The symmetries under which the equation (11) is invariant can be spanned by the following five linearly independent infinitesimal generators:

$$\begin{aligned}
&W_1 = 2VC\tfrac{\partial}{\partial V} + (C^2 - V^2)\tfrac{\partial}{\partial C},\ \ W_2 = -V\tfrac{\partial}{\partial V} - C\tfrac{\partial}{\partial C},\ \ W_3 = x^1\tfrac{\partial}{\partial x^1} + x^0\tfrac{\partial}{\partial x^0},\\
&W_4 = -\tfrac{\partial}{\partial C},\ \ W_5 = \tfrac{\partial}{\partial x^0}.
\end{aligned} \tag{26}$$

The adjoint action is given by the Lie series

$$Ad(\exp(\epsilon W_i))W_j = W_j - \epsilon[W_i, W_j] + \frac{\epsilon^2}{2}[W_i, [W_i, W_j]] - ..., \tag{27}$$

where $[W_i, W_j] = W_i W_j - W_j W_i$ is the commutator for the Lie algebra, and ϵ is a parameter. With the help of Lie series (27), the commutator table and adjoint table for Lie algebra (26) can be easily constructed as shown in the Tables 1 and 2 given below.

Table 1

Commutator Table

Index	W_1	W_2	W_3	W_4	W_5
W_1	0	$-W_1$	0	$-W_2$	0
W_2	W_1	0	0	W_4	0
W_3	0	0	0	0	$-W_5$
W_4	W_2	W_4	0	0	0
W_5	0	0	W_5	0	0

Table 2

Adjoint Table

Index	W_1	W_2	W_3	W_4	W_5
W_1	W_1	$W_2+\epsilon W_1$	$W_3+2\epsilon W_1$	$W_4+\epsilon W_2+\frac{\epsilon^2}{2}W_1$	W_5
W_2	$W_1\exp(-\epsilon)$	W_2	W_3	$W_4\exp\epsilon$	W_5
W_3	W_1	W_2	W_3	W_4	$W_5\exp(\epsilon)$
W_4	$W_1-\epsilon W_2-\frac{\epsilon^2}{2}W_4$	$W_2-\epsilon W_4$	W_3	W_4	W_5
W_5	W_1	W_2	$W_3-\epsilon W_5$	W_4	W_5

The problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. For one-dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation. This problem is attacked by the naive approach of taking a general element $W = a_1W_1 + a_2W_3 + a_3W_3 + a_4W_4 + a_5W_5$ in the Lie algebra and subjecting it to various adjoint transformations so as to simplify it as much as possible. The idea of using the adjoint representation to classify group-invariant solutions was due to Ovsiannikov [24]. We thus deduce the following basic fields, which form an optimal system for Eqs. (10) and (11):

$$\begin{aligned}
&(i)W_5\\
&(ii)W_4+\alpha W_5\\
&(iii)W_3+\beta W_4\\
&(iv)W_2+\gamma W_3\\
&(v)W_1+\delta W_3+\lambda W_4,
\end{aligned}\tag{28}$$

where $\alpha,\beta,\gamma,\delta$, and λ are arbitrary constants.

3. REDUCTIONS AND EXACT SOLUTIONS

In the following we consider, corresponding to each generator in the optimal system of sub algebras, the reductions of PDEs (10) and (11) into ODEs in terms of similarity variable ζ and the new dependent variables F and G. Some exact solutions of each reduced system are then attempted.

(i) W_5

The vector field, W_5, in the optimal system defines the similarity variable and similarity solution as follows:

$$\zeta = x^1, \;\; V(x^1,x^0) = F(\zeta), \;\; C(x^1,x^0) = G(\zeta).$$

Using the similarity variable and the forms of the similarity solution, the PDEs (10) and (11) reduce to the following system of ODEs:

$$\begin{aligned} &\zeta F F'' + F F' - \zeta F'^2 + \zeta G'^2 = 0, \\ &-\zeta F G'' - F G' + 2\zeta F' G' = 0. \end{aligned} \tag{29}$$

After solving this system of ODEs (29), we obtain the following solution of equations (10) and (11):

$$\begin{aligned} C(x^1,x^0) &= \frac{-c_2c_3+\tanh(\frac{-1}{2}\ln(x^1)\sqrt{c_1c_2}+\frac{c_4\sqrt{c_1c_2}}{2})\sqrt{c_1c_2}}{c_2}, \\ V(x^1,x^0) &= \pm\frac{\sqrt{-\zeta(2\zeta G'G'''-2\zeta G''^2+2G'G'')}G'^2}{\zeta G'G'''-\zeta G''^2+G'G''}, \end{aligned} \tag{30}$$

where $c_1,\ c_2,\ c_3$, and c_4 are arbitrary constants.

(ii) $W_4 + \alpha W_5$

For this generator, the associated similarity variable and similarity solution are as follows:

$$\zeta = x^1, \;\; V(x^1,x^0) = F(\zeta), \;\; C(x^1,x^0) = \frac{x^0}{\alpha} + G(\zeta).$$

The corresponding reduced system of ODEs is given by

$$\begin{aligned} &\alpha^2 \zeta F F'' + \alpha^2 F F' - \alpha^2 \zeta F'^2 + \alpha^2 \zeta G' 2 - \zeta = 0, \\ &\zeta F G'' + F G' - 2\zeta F' G' = 0. \end{aligned} \tag{31}$$

In this case, we are only able to produce the following solutions:

$$F(\zeta) = c_1\sqrt{\zeta}, \;\; G(\zeta) = \pm\frac{\zeta}{\alpha} + c_2, \tag{32}$$

and

$$F(\zeta) = \frac{1}{2}c_1c_3(\zeta^{(\frac{1}{\alpha c_1})}c_3^{-2} + \zeta^{(\frac{-1}{\alpha c_1})}), \;\; G(\zeta) = c_4. \tag{33}$$

Thus, the solution of PDEs (10)-(14) can be expressed as follows:

$$V(x^1,x^0) = c_1\sqrt{x^1},\ C(x^1,x^0) = \pm\frac{x^1}{\alpha} + \frac{x^0}{\alpha} + c_2,$$
$$\xi(x^1,x^0) = \frac{\mp 2x^0}{c_1^2\alpha^2} + \frac{2x^1}{c_1^2\alpha^2} + \frac{1}{4}\ln(x^1) + c_5, \tag{34}$$

and

$$V(x^1,x^0) = \frac{1}{2}c_1c_3((x^1)^{(\frac{1}{\alpha c_1})}c_3^{-2} + (x^1)^{(\frac{-1}{\alpha c_1})}),\ C(x^1,x^0) = \frac{x^0}{\alpha} + c_4,$$

$$\xi(x^1,x^0) = \frac{\frac{-c_3^2(-1+2\alpha c_1-\alpha^2c_1^2)}{\alpha c_1} - \frac{(-1+2\alpha c_1-\alpha^2c_1^2)\ln(x^1)(x^1)^{\frac{2}{\alpha c_1}}}{\alpha c_1}}{c_1\alpha((x^1)^{\frac{2}{\alpha c_1}}+c_3^2)} + \ln((x^1)^{\frac{2}{\alpha c_1}} + c_3^2) + c_4, \tag{35}$$

where $c_1,\ c_2,\ c_3$, and c_4 are arbitrary constants.

$(iii)\ \ W_3 + \beta W_4$

For this vector field, the forms of the similarity variable and the similarity solution are as follows:

$$\zeta = \frac{x^1}{x^0},\ V(x^1,x^0) = F(\zeta),\ C(x^1,x^0) = \beta\ln(x^0) + G(\zeta).$$

On substituting these forms in the PDEs (10) and (11), the reduced system of ODEs is given by

$$\zeta FF'' - \zeta^3FF'' - 2\zeta^2FF' + FF' - \zeta F^2 + \zeta^3F'^2 + 2\beta\zeta^2G' - \zeta^3G'^2$$
$$+\zeta G'^2 - \beta^2\zeta = 0,$$
$$-\zeta FG'' - \beta\zeta F + \zeta^3FG'' + 2\zeta^2FG' - FG' + 2\zeta F'G' + 2\beta\zeta^2F' - 2\zeta^3F'G' = 0, \tag{36}$$

which is quite difficult to solve. Therefore by taking $\beta = 0$, the ODEs (36) become

$$\zeta FF'' - \zeta^3FF'' - 2\zeta^2FF' + FF' - \zeta F^2 + \zeta^3F'^2 - \zeta^3G'^2 + \zeta G'^2 = 0,$$
$$-\zeta FG'' + \zeta^3FG'' + 2\zeta^2FG' - FG' + 2\zeta F'G' - 2\zeta^3F'G' = 0. \tag{37}$$

which can be further solved to give the following solution

$$F(\zeta) = c_1\sqrt{\zeta}(\zeta^2-1)^{\frac{1}{4}}\sqrt{\left(\frac{1}{2\zeta\sqrt{\zeta^2-1}c_2^2\left(\cosh\left(\frac{arctan(\frac{1}{\sqrt{\zeta^2-1}})+c_3}{c_1c_2}\right)+1\right)}\right)},$$
$$G(\zeta) = \int\left(\frac{1}{2\zeta\sqrt{\zeta^2-1}c_2^2\left(\cosh\left(\frac{\arctan(\frac{1}{\sqrt{\zeta^2-1}})+c_3}{c_1c_2}\right)+1\right)}\right)d\zeta + c_4, \tag{38}$$

where $c_1,\ c_2,\ c_3$, and c_4 are arbitrary constants.

Using these results we get final solution of Eqs. (10) and (11):

$$V(x^1,x^0) = c_1\sqrt{(\tfrac{x^1}{x^0})}((\tfrac{x^1}{x^0})^2-1)^{\frac{1}{4}}\times \sqrt{\left(\frac{1}{2(\frac{x^1}{x^0})\sqrt{(\frac{x^1}{x^0})^2-1}c_2^2\left(\cosh\left(\frac{arctan(\frac{1}{\sqrt{(\frac{x^1}{x^0})^2-1}})+c_3}{c_1c_2}\right)+1\right)}\right)},$$
$$C(x^1,x^0) = \int\left(\frac{1}{2\zeta\sqrt{\zeta^2-1}c_2^2\left(\cosh(\frac{\arctan(\frac{1}{\sqrt{\zeta^2-1}})+c_3}{c_1c_2})+1\right)}\right)d\zeta + c_4. \tag{39}$$

$(iv)\ \ W_2+\gamma W_3$

In this case of vector field in the optimal system, we obtain

$$\zeta = \frac{x^1}{x^0},\ \ V(x^1,x^0) = (x^1)^{(\frac{-1}{\gamma})}F(\zeta),\ \ C(x^1,x^0) = (x^1)^{(\frac{-1}{\gamma})}G(\zeta).$$

For the case under consideration the reduced ODEs are

$$\begin{aligned}&\gamma^2FF''-\zeta^2\gamma^2FF''-2\zeta\gamma^2FF'+\tfrac{\gamma^2FF'}{\zeta}-\gamma^2F'^2+\zeta^2\gamma^2F'^2-\zeta^2\gamma^2G'^2+\\&\tfrac{-2\gamma GG'}{\zeta}+\gamma^2G'^2+\tfrac{G^2}{\zeta^2}=0,\\&-\tfrac{FG}{\zeta^2}+\gamma^2G''F-\zeta^2\gamma^2FG''-2\zeta\gamma^2FG'+\tfrac{\gamma^2FG'}{\zeta}+\tfrac{2\gamma GF'}{\zeta}-2\gamma^2G'F'+\\&2\zeta^2\gamma^2F'G'=0.\end{aligned} \tag{40}$$

The solution of this system of ODEs (40) is as follow:
Let $G(\zeta) = \pm\iota F(\zeta)$. Using these substitutions our system reduces to the following single equation

$$\begin{aligned}&\zeta^2\gamma^2FF''-\zeta^4\gamma^2FF''-2\zeta^3\gamma^2FF'+\zeta\gamma^2FF'-2\zeta^2\gamma^2F'^2+2\zeta^4\gamma^2F'^2+\\&2\zeta\gamma FF'-F^2=0,\end{aligned} \tag{41}$$

hence we arrive at the following solution:

$$F(\zeta) = A/B\,, \tag{42}$$

where A and B are, respectively

$$-c_1\exp(\int\frac{(-(-1+\alpha)\zeta^2\mathrm{hypergeom}\,(p,(-\zeta^2+1))+\mathrm{hypergeom}\,(q,-\zeta^2+1)(\alpha-2))}{\mathrm{hypergeom}\,(r,\zeta^2-1)(\alpha-2)\alpha\zeta}d\zeta),$$

$$\int((\frac{(\zeta^2-1)^{\frac{(2-\alpha)}{2\alpha}}}{\zeta})\exp(\frac{-2}{\alpha(\alpha-2)}((\alpha-1)\int(\frac{\zeta^3}{\zeta^2-1}\frac{\mathrm{hypergeom}\,(p,-\zeta^2+1)}{\mathrm{hypergeom}\,(q,-\zeta^2+1)}d\zeta)+(1-\alpha)\int(\frac{\zeta}{\zeta^2-1}\frac{\mathrm{hypergeom}\,(p,-\zeta^2+1)}{\mathrm{hypergeom}\,(q,-\zeta^2+1)}d\zeta))))d\zeta$$

and $p = [\frac{-1+2\alpha}{2\alpha}, \frac{-1+3\alpha}{2\alpha}], [\frac{3\alpha-2}{2\alpha}]$, $q = [\frac{-1}{2\alpha}, \frac{-1+\alpha}{2\alpha}], [\frac{\alpha-2}{2\alpha}]$, and c_1 is an arbitrary constant. Making substitutions of these expressions for $F(\zeta)$, the solution of Eqs. (10) and (11) are furnished as follows:

$$\begin{aligned} V(x^1, x^0) &= (x^1)^{(\frac{-1}{\gamma})} F(\zeta), \\ C(x^1, x^0) &= \pm\iota (x^1)^{(\frac{-1}{\gamma})} F(\zeta). \end{aligned} \tag{43}$$

(v) $W_1 + \delta W_3 + \lambda W_4$

Unfortunately, we are not able to reduce the ODEs corresponding to this case; this will be taken up in a future endeavor. We consider the simplest form of transformation in view of the above similarity variables and similarity functions as follows:

$$C = \pm \exp(V).$$

Using this in Eqs. (10) and (11), we obtain

$$x^1 V V_{11} - x^1 V V_{00} + V V_1 - 2x^1 V_1^2 + 2x^1 V_0^2 = 0. \tag{44}$$

and the solution is

$$\begin{aligned} V(x^1, x^0) &= \frac{x^1 c_1^3 (J_1(c_1 x^1) Y_0(c_1 x^1) - J_0(c_1 x^1) Y_1(c_1 x^1))}{(-c_3 J_0(c_1 x^1) + c_4 Y_0(c_1 x^1))(c_3 \sin(c_1 x^0) - c_4 \cos(c_1 x^0))}, \\ C(x^1, x^0) &= \pm\iota \frac{x^1 c_1^3 (J_1(c_1 x^1) Y_0(c_1 x^1) - J_0(c_1 x^1) Y_1(c_1 x^1))}{(-c_3 J_0(c_1 x^1) + c_4 Y_0(c_1 x^1))(c_3 \sin(c_1 x^0) - c_4 \cos(c_1 x^0))} \end{aligned} \tag{45}$$

where c_1, c_3, and c_4 are arbitrary constants and $J_v(x)$ and $Y_v(x)$ are the modified Bessel functions of the first and second kinds, respectively. They satisfy the modified Bessel equation:

$$x^2 Y'' + x Y' - (x^2 + v^2) Y = 0.$$

Using the above expressions for $V(x^1, x^0)$ and $C(x^1, x^0)$ in Eqs. (12)-(14) and solving them, we get

$$\xi(x^1, x^0) = c_5, \tag{46}$$

where c_5 is an arbitrary constant.

4. CONCLUDING REMARKS

We investigated the field equations of general relativity with an electromagnetic stress tensor as source and Maxwell's equations in curved space. We have utilized the symmetry method based on the Fréchet derivative of the differential operators to obtain the Lie symmetries admitted by these equations. The metric coefficients and electromagnetic fields are restricted to be functions of two independent variables only. The infinitesimal generators in optimal system of sub-algebras of the full Lie algebra of the coupled system of nonlinear partial differential equations of second order of field equations are considered. We completely solved the determining equations for the infinitesimal generators of Lie groups. Further, the group classification from the point of view of the optimal system of non-conjugate sub-algebras of the symmetry algebra of the nonlinear system has been performed under the adjoint action of the symmetry group. The various fields in the optimal system have been then exploited to get the reductions of PDEs into ODEs. Some exact solutions are attempted for the reduced systems that might prove to be interesting for further applications. The obtained results are new and the analysis is reliable. It is hoped that the obtained results will enrich the theories for the associated nonlinear equations describing electromagnetic fields.

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