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DIFFUSION PROBLEMS IN COMPOSITE MEDIA WITH INTERFACIAL FLUX JUMP

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Abstract. We study the homogenization of a thermal diffusion problem in a highly heterogeneous composite medium formed by two constituents, separated by an imperfect interface, where both the temperature and the flux exhibit jumps. Two geometrical settings are considered, in terms of the connectivity of the two constituents. The presence of the flux jump leads to some modified stationary Barenblatt models.

Key words: homogenization, imperfect interfaces, Barenblatt model.

1. INTRODUCTION

In this paper, we analyze, *via* the periodic unfolding method, the effective thermal diffusion in a periodic composite material formed by two constituents occupying a domain Ω in $\mathbb{R}^N(N \geq 2)$, divided in two open subdomains, Ω_1^ε and Ω_2^ε, separated by an imperfect interface Γ^ε. Here, ε is a real positive parameter less than one, related to the characteristic dimension of the two subdomains. From the point of view of the geometry of the domain, we study two cases. In both ones, the phase Ω_1^ε is connected, while the phase Ω_2^ε is disconnected in the first situation and connected in the second one. This last situation can occur only for $N \geq 3$. We shall study, in these geometrical settings, the asymptotic behavior, as the small parameter ε tends to zero, of the solution $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ of problems (2.1) and (4.1), respectively. The main feature of these two problems is the presence of a jump in the solution and in its flux, as well, across the imperfect interface Γ^ε. We shall also compare the effect produced, at the macroscale, by the diffusion process taking place in the component Ω_2^ε of the composite structure in the two geometrical settings mentioned above.

The mathematical study of thermal problems involving a continuous flux (*i.e.* the case $g^\varepsilon = 0$ in (2.1) and (4.1)) which is proportional to the jump of the solution is well-understood by now (see, *e.g.*, [1–10]). For transmission problems in other contexts, see [11–16]. The case of a discontinuous flux has been only recently addressed in [17–22]. We notice that if, instead of dealing with a two-component domain, we consider a classical perforated domain Ω_1^ε, we are led to a Robin problem. The literature dedicated to the homogenization of such problems is vast (we refer, for instance, to [23–29] and the references therein).

By passing to the limit in (2.1) and (4.1), *via* the periodic unfolding method (see [1, 5, 30, 31]), we get, at the macroscale, some modified stationary Barenblatt models. Various microscopic problems can lead to the Barenblatt model introduced in [32] (see [33] for physical and mathematical aspects of such models). For rigorous mathematical justifications of such a model, we refer, *e.g.*, to [34, 35] for the connected-disconnected case and to [3, 36, 37] for the connected-connected one.

In contrast to the above cited papers, we consider in our microscopic problem a jump in the flux of the solution and it is exactly the presence of this jump that leads to an additional term in the classical form of such Barenblatt models.

The paper is organized as follows: in Section 2, we set the microscopic problem in the connected-disconnected case and we prove the corresponding homogenization result in Section 3. In Section 4, we prove the convergence result in the connected-connected case. The paper ends with a few conclusions and some references.

2. SETTING OF THE PROBLEM IN THE CONNECTED–DISCONNECTED CASE

Let Ω be a bounded open set in $\mathbb{R}^N$ $(N \geq 2)$, with a Lipschitz continuous boundary $\partial\Omega$ and let $Y = (0,1)^N$ be the reference cell in $\mathbb{R}^N$. We assume that Y_1 and Y_2 are two non-empty disjoint connected open subsets of Y such that $\overline{Y}_2 \subset Y$ and $Y = Y_1 \cup \overline{Y}_2$. We also suppose that $\Gamma = \partial Y_2$ is Lipschitz continuous. For each $k \in \mathbb{Z}^N$, we denote $Y^k = k + Y$ and $Y_\alpha^k = k + Y_\alpha$, for $\alpha \in \{1,2\}$.

Let ε be a small parameter taking values in a positive real sequence tending to zero. We define, for each ε, $\mathbb{Z}_\varepsilon = \left\{ k \in \mathbb{Z}^N \,|\, \varepsilon \overline{Y_2^k} \subset \Omega \right\}$ and we set $\Omega_2^\varepsilon = \bigcup_{k \in \mathbb{Z}_\varepsilon} \left(\varepsilon Y_2^k \right)$ and $\Omega_1^\varepsilon = \Omega \setminus \overline{\Omega_2^\varepsilon}$. The boundary of Ω_2^ε is denoted by Γ^ε and n^ε is the unit outward normal to Ω_2^ε.

Our goal in this paper is to analyze the asymptotic behavior, as $\varepsilon \to 0$, of the solution $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ of the following problem:

$$\begin{cases} -\text{div}\,(A^\varepsilon \nabla u_1^\varepsilon) = f & \text{in } \Omega_1^\varepsilon, \\ -\text{div}\,(A^\varepsilon \nabla u_2^\varepsilon) = f & \text{in } \Omega_2^\varepsilon, \\ A^\varepsilon \nabla u_1^\varepsilon \cdot n^\varepsilon = \varepsilon h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon) - \varepsilon g^\varepsilon & \text{on } \Gamma^\varepsilon, \\ A^\varepsilon \nabla u_2^\varepsilon \cdot n^\varepsilon = \varepsilon h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon) & \text{on } \Gamma^\varepsilon, \\ u_1^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Remark 2.1 *The flux of the solution is discontinuous across Γ^ε. Indeed, we have*

$$A^\varepsilon \nabla u_1^\varepsilon \cdot n^\varepsilon - A^\varepsilon \nabla u_2^\varepsilon \cdot n^\varepsilon = -\varepsilon g^\varepsilon. \tag{2.2}$$

The function $f \in L^2(\Omega)$ is given. Further, we assume that:

(H1) For $\lambda, \mu \in \mathbb{R}$, with $0 < \lambda \leq \mu$, let $\mathcal{M}(\lambda, \mu, Y)$ be the set of all the matrices $A \in (L^\infty(Y))^{N\times N}$ such that for any $\xi \in \mathbb{R}^N$, $\lambda|\xi|^2 \leq (A(y)\xi, \xi) \leq \mu|\xi|^2$, a.e. in Y. For a Y-periodic symmetric matrix $A \in \mathcal{M}(\lambda, \mu, Y)$, we define

$$A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right) \qquad \text{a.e. in } \Omega.$$

(H2) h is a Y–periodic function such that $h \in L^\infty(\Gamma)$ and there exists $h_0 \in \mathbb{R}$ with $0 < h_0 < h(y)$ a.e. on Γ. We set

$$h^\varepsilon(x) = h\left(\frac{x}{\varepsilon}\right) \qquad \text{a.e. on } \Gamma^\varepsilon.$$

(H3) g is a Y–periodic function that belongs to $L^2(\Gamma)$ and such that its mean value over Γ, $\mathcal{M}_\Gamma(g) = \dfrac{1}{|\Gamma|}\displaystyle\int_\Gamma g(y)\,\mathrm{d}y$, is not zero. We put

$$g^\varepsilon(x) = g\left(\frac{x}{\varepsilon}\right) \qquad \text{a.e. on } \Gamma^\varepsilon.$$

We introduce, for $\varepsilon \in (0,1)$, the Hilbert space $H^\varepsilon = V^\varepsilon \times H^1(\Omega_2^\varepsilon)$. On the space H^ε, we consider the scalar product

$$(u,v)_{H^\varepsilon} = \int_{\Omega_1^\varepsilon} \nabla u_1 \nabla v_1\,\mathrm{d}x + \int_{\Omega_2^\varepsilon} \nabla u_2 \nabla v_2\,\mathrm{d}x + \varepsilon\int_{\Gamma^\varepsilon} (u_1 - u_2)(v_1 - v_2)\,\mathrm{d}\sigma_x, \quad (2.3)$$

where $u = (u_1, u_2)$ and $v = (v_1, v_2)$ belong to H^ε.

The variational formulation of problem (2.1) writes: find $u^\varepsilon \in H^\varepsilon$ such that

$$a(u^\varepsilon, v) = l(v), \quad \forall v \in H^\varepsilon, \quad (2.4)$$

with the forms $a : H^\varepsilon \times H^\varepsilon \to \mathbb{R}$ and $l : H^\varepsilon \to \mathbb{R}$ given by

$$a(u,v) = \int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1 \nabla v_1\,\mathrm{d}x + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_2 \nabla v_2\,\mathrm{d}x + \varepsilon\int_{\Gamma^\varepsilon} h^\varepsilon (u_1 - u_2)(v_1 - v_2)\,\mathrm{d}\sigma_x,$$

$$l(v) = \int_{\Omega_1^\varepsilon} f v_1\,\mathrm{d}x + \int_{\Omega_2^\varepsilon} f v_2\,\mathrm{d}x + \varepsilon\int_{\Gamma^\varepsilon} g^\varepsilon v_1\,\mathrm{d}\sigma_x.$$

We state now an existence and uniqueness result and some *a priori* estimates for the solution of the variational problem (2.4). Here and throughout the paper, C is a positive constant independent of ε, whose value can change from line to line.

Theorem 2.2 *For any* $\varepsilon \in (0,1)$*, the variational problem* (2.4) *has a unique solution* $u^\varepsilon \in H^\varepsilon$*. Besides, there exists a constant* C *such that*

$$\|u_1^\varepsilon\|_{L^2(\Omega_1^\varepsilon)} \leq C, \quad \|u_2^\varepsilon\|_{L^2(\Omega_2^\varepsilon)} \leq C, \quad (2.5)$$

$$\|\nabla u_1^\varepsilon\|_{L^2(\Omega_1^\varepsilon)} \leq C, \quad \|\nabla u_2^\varepsilon\|_{L^2(\Omega_2^\varepsilon)} \leq C, \quad \varepsilon^{1/2}\|u_1^\varepsilon - u_2^\varepsilon\|_{L^2(\Gamma^\varepsilon)} \leq C. \quad (2.6)$$

Proof. The proof of the existence and uniqueness result can be done by using the Lax-Milgram theorem. For the continuity and the coercivity of the bilinear form a, we refer to [1]. For the continuity of l, we refer to Theorem 2.3 in [19]. Relations (2.5) and (2.6) are then obtained as in [1]. ▪

3. HOMOGENIZATION RESULTS FOR THE CONNECTED–DISCONNECTED CASE

Our goal in this section is to get the effective behavior of the solution of problem (2.1). To this end, we will pass to the limit, when ε tends to zero, in the variational formulation (2.4) by using the periodic unfolding method and the general compactness results given in [1]. According to the *a priori* estimates (2.5)-(2.6), the compactness results from [1] for bounded sequences in the space H^ε imply that there exist $u_1 \in H_0^1(\Omega)$, $\widehat{u}_1 \in L^2(\Omega, H^1_{\text{per}}(Y_1))$, $u_2 \in L^2(\Omega)$, $\widehat{u}_2 \in L^2(\Omega, H^1(Y_2))$ such that $\mathcal{M}_\Gamma(\widehat{u}_1) = 0$, $\mathcal{M}_\Gamma(\widehat{u}_2) = 0$ and, up to a subsequence, for $\varepsilon \to 0$, we have:

$$\begin{array}{l} \mathcal{T}_1^\varepsilon(u_1^\varepsilon) \to u_1 \quad \text{strongly in } L^2(\Omega, H^1(Y_1)), \\ \mathcal{T}_1^\varepsilon(\nabla u_1^\varepsilon) \rightharpoonup \nabla u_1 + \nabla_y \widehat{u}_1 \quad \text{weakly in } L^2(\Omega \times Y_1), \\ \mathcal{T}_2^\varepsilon(u_2^\varepsilon) \rightharpoonup u_2 \quad \text{weakly in } L^2(\Omega, H^1(Y_2)), \\ \mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon) \rightharpoonup \nabla_y \widehat{u}_2 \quad \text{weakly in } L^2(\Omega \times Y_2), \\ \widetilde{u}_1^\varepsilon \rightharpoonup |Y_1| u_1 \quad \text{weakly in } L^2(\Omega), \\ \widetilde{u}_2^\varepsilon \rightharpoonup |Y_2| u_2 \quad \text{weakly in } L^2(\Omega), \end{array} \tag{3.1}$$

where by $\widetilde{\cdot}$ we denote the extension by zero of a function to the whole of Ω and $H^1_{\text{per}}(Y_1) = \{v \in H^1(Y_1) \mid v \text{ is Y-periodic}\}$. Moreover, we define the spaces

$$W_{\text{per}}(Y_1) = \{v \in H^1_{\text{per}}(Y_1) \mid \mathcal{M}_\Gamma(v) = 0\}, \; W(Y_2) = \{v \in H^1(Y_2) \mid \mathcal{M}_\Gamma(v) = 0\},$$

$$\mathcal{V} = H_0^1(\Omega) \times L^2(\Omega, W_{\text{per}}(Y_1)) \times L^2(\Omega) \times L^2(\Omega, W(Y_2)).$$

In (3.1), we omitted to write $|Y|$. Since $|Y| = 1$, in the sequel we shall not write it.

Theorem 3.1 *The unique solution $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ of the variational problem (2.4) converges, in the sense of (3.1), to $(u_1, \widehat{u}_1, u_2, \widehat{u}_2) \in \mathcal{V}$, such that the couple $(u_1, \widehat{u}_1)$ is the unique solution of the following unfolded limit problem:*

$$\int_{\Omega \times Y_1} A(y)(\nabla u_1 + \nabla_y \widehat{u}_1)(\nabla \varphi_1 + \nabla_y \Phi_1)\, dx\, dy =$$

$$\int_\Omega f(x)\varphi_1(x)\, dx + |\Gamma| \mathcal{M}_\Gamma(g) \int_\Omega \varphi_1(x)\, dx, \tag{3.2}$$

for all $\varphi_1 \in H_0^1(\Omega)$, $\Phi_1 \in L^2(\Omega, H^1_{per}(Y_1))$,

$$u_2(x) = u_1(x) + \frac{|Y_2|}{|\Gamma| \mathcal{M}_\Gamma(h)} f(x) \quad \textit{in } \Omega, \tag{3.3}$$

and

$$\widehat{u}_2(x,y) = 0 \quad \textit{in } \Omega\times Y_2. \tag{3.4}$$

Proof. For ε small enough, by unfolding the variational formulation (2.4), we obtain (see [30]):

$$\int_{\Omega\times Y_1} \mathcal{T}_1^\varepsilon(A^\varepsilon)\mathcal{T}_1^\varepsilon(\nabla u_1^\varepsilon)\mathcal{T}_1^\varepsilon(\nabla v_1)\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega\times Y_2} \mathcal{T}_2^\varepsilon(A^\varepsilon)\mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon)\mathcal{T}_2^\varepsilon(\nabla v_2)\,\mathrm{d}x\,\mathrm{d}y +$$

$$\int_{\Omega\times\Gamma} h(y)(\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - \mathcal{T}_2^\varepsilon(u_2^\varepsilon))(\mathcal{T}_1^\varepsilon(v_1) - \mathcal{T}_2^\varepsilon(v_2))\,\mathrm{d}x\,\mathrm{d}\sigma_y \simeq$$

$$\int_{\Omega\times Y_1} \mathcal{T}_1^\varepsilon(f)\mathcal{T}_1^\varepsilon(v_1)\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega\times Y_2} \mathcal{T}_2^\varepsilon(f)\mathcal{T}_2^\varepsilon(v_2)\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega\times\Gamma} \mathcal{T}_b^\varepsilon(g^\varepsilon)\mathcal{T}_b^\varepsilon(v_1)\,\mathrm{d}x\,\mathrm{d}\sigma_y,$$

for all $(v_1,v_2)\in H^\varepsilon$. In this unfolded problem, we choose the test functions

$$v_1 = 0, \quad v_2 = \varepsilon\omega_2(x)\psi_2\left(\frac{x}{\varepsilon}\right),$$

with $\omega_2\in\mathcal{D}(\Omega)$, $\psi_2\in H^1(Y_2)$ (we suppose that, for $\alpha\in\{1,2\}$, any function defined on Y_α is extended by Y-periodicity to the whole of $\mathbb{R}^N$). Passing to the limit and then using the density of $\mathcal{D}(\Omega)\otimes H^1(Y_2)$ in $L^2(\Omega,H^1(Y_2))$, we obtain:

$$\int_{\Omega\times Y_2} A(y)\nabla_y\widehat{u}_2\nabla_y\Phi_2\,\mathrm{d}x\,\mathrm{d}y = 0, \quad \forall\Phi_2\in L^2(\Omega,H^1(Y_2)), \tag{3.5}$$

which, as in [1, 21], leads to $\nabla_y\widehat{u}_2 = 0$. Using now the fact that $\mathcal{M}_\Gamma(\widehat{u}_2) = 0$, we apply the Poincaré-Wirtinger inequality to the function $\widehat{u}_2$ in $H^1(Y_2)$ and we get $||\widehat{u}_2(x,y)||_{L^2(\Omega\times Y_2)} = 0$, which implies (3.4).

Choosing now in the unfolded problem the test functions

$$v_1 = \varphi_1(x) + \varepsilon\omega_1(x)\psi_1\left(\frac{x}{\varepsilon}\right), \quad v_2 = \varphi_2(x) + \varepsilon\omega_2(x)\psi_2\left(\frac{x}{\varepsilon}\right), \tag{3.6}$$

with $\varphi_1,\varphi_2,\omega_1,\omega_2\in\mathcal{D}(\Omega)$, $\psi_1\in H^1_{\text{per}}(Y_1)$, $\psi_2\in H^1(Y_2)$, it is not difficult to see that we have the following convergences:

$$\mathcal{T}_1^\varepsilon(v_1)\to\varphi_1 \quad \text{strongly in } L^2(\Omega\times Y_1), \tag{3.7}$$

$$\mathcal{T}_1^\varepsilon(\nabla v_1)\to\nabla\varphi_1 + \nabla_y\Phi_1 \quad \text{strongly in } L^2(\Omega\times Y_1), \tag{3.8}$$

$$\mathcal{T}_2^\varepsilon(v_2)\to\varphi_2 \quad \text{strongly in } L^2(\Omega\times Y_2), \tag{3.9}$$

$$\mathcal{T}_2^\varepsilon(\nabla v_2)\to\nabla\varphi_2 + \nabla_y\Phi_2 \quad \text{strongly in } L^2(\Omega\times Y_2), \tag{3.10}$$

where $\Phi_1(x,y) = \omega_1(x)\psi_1(y)$ and $\Phi_2(x,y) = \omega_2(x)\psi_2(y)$.

The passage to the limit in the unfolded problem is standard, by using (3.4), (3.1) and (3.7)-(3.10) (see, *e.g.*, [1, 30]). Then, by the density of $\mathcal{D}(\Omega)\otimes H^1_{\text{per}}(Y_1)$ in $L^2(\Omega, H^1_{\text{per}}(Y_1))$ and of $\mathcal{D}(\Omega)\otimes H^1(Y_2)$ in $L^2(\Omega, H^1(Y_2))$, we obtain

$$\int_{\Omega\times Y_1} A(y)(\nabla u_1 + \nabla_y \widehat{u}_1)(\nabla\varphi_1 + \nabla_y \Phi_1)\,\mathrm{d}x\,\mathrm{d}y +$$

$$|\Gamma|\mathcal{M}_\Gamma(h)\int_\Omega (u_1-u_2)(\varphi_1-\varphi_2)\,\mathrm{d}x =$$

$$\int_{\Omega\times Y_1} f(x)\varphi_1(x)\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega\times Y_2} f(x)\varphi_2(x)\,\mathrm{d}x\,\mathrm{d}y + |\Gamma|\mathcal{M}_\Gamma(g)\int_\Omega \varphi_1(x)\,\mathrm{d}x. \quad (3.11)$$

We choose now $\varphi_1 = 0$ and $\Phi_1 = 0$ in (3.11) and we obtain

$$-|\Gamma|\mathcal{M}_\Gamma(h)(u_1-u_2) = |Y_2|\, f,\ \text{ in } \Omega, \quad (3.12)$$

which implies (3.3). Replacing (3.12) in (3.11), we obtain (3.2).

The well-posedness of problem (3.2) follows from Lax-Milgram theorem applied in the space $H^1_0(\Omega)\times L^2(\Omega, W_{\text{per}}(Y_1))$, endowed with the norm $\|\nabla v(x) + \nabla_y \widehat{v}(x,y)\|_{L^2(\Omega\times Y_1)}$. Due to the uniqueness of $(u_1,\widehat{u}_1)\in H^1_0(\Omega)\times L^2(\Omega, W_{\text{per}}(Y_1))$, all the above convergences hold true for the whole sequence. ▪

Theorem 3.2 *The unique solution $(u_1,\widehat{u}_1)$ of problem (3.2) is such that*

$$-div\,(A_1^{hom}\nabla u_1(x)) = f(x) + |\Gamma|\mathcal{M}_\Gamma(g) \quad in\ \Omega, \quad (3.13)$$

$$\widehat{u}_1(x,y) = -\sum_{j=1}^{N} \frac{\partial u_1}{\partial x_j}(x)\chi_1^j(y) \quad in\ \Omega\times Y_1. \quad (3.14)$$

Here, the positive definite homogenized matrix A_1^{hom} is given, for $i,j = 1,\dots,N$, by

$$A^{hom}_{1,ij} = \int_{Y_1}\left(a_{ij} - \sum_{k=1}^{N} a_{ik}\frac{\partial\chi_1^j}{\partial y_k}\right)\mathrm{d}y \quad (3.15)$$

and the j-th component of the vectorial function χ_1, $\chi_1^j \in H^1_{per}(Y_1)$ ($j = 1,\dots,N$), is the unique weak solution of the following cell problem:

$$\begin{cases} -div_y(A(y)(\nabla_y\chi_1^j - e_j)) = 0, & in\ Y_1,\\ (A(y)(\nabla_y\chi_1^j - e_j))\cdot n = 0, & on\ \Gamma,\\ \mathcal{M}_\Gamma(\chi_1^j) = 0, & \end{cases} \quad (3.16)$$

where n denotes the unit normal interior to Y_1.

Proof. By choosing $\varphi_1 = 0$ in the unfolded limit problem (3.2), we obtain:

$$\int_{\Omega\times Y_1} A(y)(\nabla u_1 + \nabla_y\widehat{u}_1)\nabla_y\Phi_1\,\mathrm{d}x\,\mathrm{d}y = 0, \quad \forall \Phi_1 \in L^2(\Omega, H^1_{\text{per}}(Y_1)).$$

Therefore,

$$-\mathrm{div}_y(A(y)(\nabla u_1 + \nabla_y \widehat{u}_1)) = 0 \quad \text{in } \Omega \times Y_1,$$

$$A(y)(\nabla u_1 + \nabla_y \widehat{u}_1) \cdot n = 0 \quad \text{on } \Omega \times \Gamma.$$

Then, classical results from the theory of homogenization imply (3.14) and (3.16).

We choose now $\Phi_1 = 0$ in (3.2) and we obtain

$$\int_{\Omega \times Y_1} A(y)(\nabla u_1 + \nabla_y \widehat{u}_1) \nabla \varphi_1 \, \mathrm{d}x \, \mathrm{d}y =$$

$$\int_{\Omega} f(x) \varphi_1(x) \, \mathrm{d}x + |\Gamma| \mathcal{M}_\Gamma(g) \int_{\Omega} \varphi_1(x) \, \mathrm{d}x, \ \forall \varphi_1 \in \mathcal{D}(\Omega),$$

which, together with (3.14) and (3.16), imply (3.13). ▪

Remark 3.3 *Due to the right scaling ε in front of the function g^ε given at the interface Γ^ε, the presence of the flux jump leads to the appearance in the homogenized equation* (3.13) *of an additional source term distributed all over the domain Ω and depending on the function g.*

Remark 3.4 *We remark that the limit problems* (3.13) *and* (3.3) *verified by* (u_1, u_2) *can be written in the form of a coupled system, formed by a partial differential equation and an algebraic one, and which is a modified stationary Barenblatt model:*

$$\begin{cases} -div(A_1^{hom} \nabla u_1) + |\Gamma| \mathcal{M}_\Gamma(h)(u_1 - u_2) = |Y_1| f + |\Gamma| \mathcal{M}_\Gamma(g) & \text{in } \Omega, \\ -|\Gamma| \mathcal{M}_\Gamma(h)(u_1 - u_2) = |Y_2| f & \text{in } \Omega, \\ u_1(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.17}$$

4. THE CONNECTED–CONNECTED CASE

In this section, we shall deal with the case in which the two constituents Ω_ε^1 and Ω_ε^2, respectively, are both connected and reach the exterior boundary $\partial\Omega$. The existence of such a geometry asks for an additional restriction, namely $N \geq 3$. More precisely, we suppose that Y_1 and Y_2 are two disjoint connected open subsets of the unit cell $Y = (0,1)^N$, with a common boundary Γ and such that both reach the boundary ∂Y of the elementary cell Y. We set $\partial Y_1 = \Gamma \cup \Gamma_1$ and $\partial Y_2 = \Gamma \cup \Gamma_2$, where Γ_α, for $\alpha \in \{1,2\}$, are the intersections of ∂Y_α with ∂Y. We suppose that Γ_α are identically reproduced on the opposite faces of Y. Also, for each $k \in \mathbb{Z}^N$, we denote, as in Section 2, $Y_\alpha^k = k + Y_\alpha$, for $\alpha \in \{1,2\}$. For each ε, we define $\mathbb{Z}_\varepsilon = \left\{ k \in \mathbb{Z}^N \,|\, \varepsilon Y_\alpha^k \cap \Omega \neq \emptyset, \ \alpha \in \{1,2\} \right\}$ and we set $\Omega_2^\varepsilon = \Omega \setminus \bigcup_{k \in \mathbb{Z}_\varepsilon} \left(\varepsilon \overline{Y_1^k} \right)$, $\Omega_1^\varepsilon = \Omega \setminus \overline{\Omega_2^\varepsilon}$ and $\Gamma^\varepsilon = \partial\Omega_1^\varepsilon \cap \Omega$. Such a geometry was considered, *e.g.*, in [5, 36].

Our goal in this section is to analyze the asymptotic behavior, as $\varepsilon \to 0$, of the solution $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ of the following problem:

$$\begin{cases} -\text{div}\,(A^\varepsilon \nabla u_1^\varepsilon) = f \quad \text{in } \Omega_1^\varepsilon, \\ -\text{div}\,(A^\varepsilon \nabla u_2^\varepsilon) = f \quad \text{in } \Omega_2^\varepsilon, \\ A^\varepsilon \nabla u_1^\varepsilon \cdot n^\varepsilon = \varepsilon h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon) - \varepsilon g^\varepsilon \quad \text{on } \Gamma^\varepsilon, \\ A^\varepsilon \nabla u_2^\varepsilon \cdot n^\varepsilon = \varepsilon h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon) \quad \text{on } \Gamma^\varepsilon, \\ u_\alpha^\varepsilon = 0 \quad \text{on } \partial\Omega \cap \partial\Omega_\alpha^\varepsilon, \quad \alpha \in \{1,2\}. \end{cases} \tag{4.1}$$

We introduce, for $\varepsilon \in (0,1)$, the new Hilbert space, still denoted by H^ε, defined by $H^\varepsilon = V_1^\varepsilon \times V_2^\varepsilon$, where $V_\alpha^\varepsilon = \{ v \in H^1(\Omega_\alpha^\varepsilon) \,|\, v = 0 \text{ on } \partial\Omega \cap \partial\Omega_\alpha^\varepsilon \}$, for $\alpha \in \{1,2\}$. The weak form of problem (4.1) is exactly (2.4) written for the new space H^ε endowed with the corresponding scalar product (2.3). Theorem 2.2 still holds true in this case.

In this new geometrical setting, the corresponding compactness results for bounded sequences in H^ε (see, for instance, [5]) imply that there exist $u_1 \in H_0^1(\Omega)$, $\widehat{u}_1 \in L^2(\Omega, H^1_{\text{per}}(Y_1))$, $u_2 \in H_0^1(\Omega)$, $\overline{u}_2 \in L^2(\Omega, H^1_{\text{per}}(Y_2))$, such that $\mathcal{M}_\Gamma(\widehat{u}_1) = 0$, $\mathcal{M}_\Gamma(\overline{u}_2) = 0$ and, up to a subsequence, for $\varepsilon \to 0$, we have:

$$\begin{aligned} &\mathcal{T}_1^\varepsilon(u_1^\varepsilon) \to u_1 \quad \text{strongly in } L^2(\Omega, H^1(Y_1)), \\ &\mathcal{T}_1^\varepsilon(\nabla u_1^\varepsilon) \rightharpoonup \nabla u_1 + \nabla_y \widehat{u}_1 \quad \text{weakly in } L^2(\Omega \times Y_1), \\ &\mathcal{T}_2^\varepsilon(u_2^\varepsilon) \to u_2 \quad \text{strongly in } L^2(\Omega, H^1(Y_2)), \\ &\mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon) \rightharpoonup \nabla u_2 + \nabla_y \overline{u}_2 \quad \text{weakly in } L^2(\Omega \times Y_2), \\ &\widetilde{u_1^\varepsilon} \rightharpoonup |Y_1| u_1 \quad \text{weakly in } L^2(\Omega), \\ &\widetilde{u_2^\varepsilon} \rightharpoonup |Y_2| u_2 \quad \text{weakly in } L^2(\Omega), \end{aligned} \tag{4.2}$$

where $H^1_{\text{per}}(Y_2) = \{ v \in H^1(Y_2) \mid v \text{ is Y-periodic} \}$. Besides, we define

$$W_{\text{per}}(Y_2) = \{ v \in H^1_{\text{per}}(Y_2) \mid \mathcal{M}_\Gamma(v) = 0 \},$$

$$\mathcal{W} = H_0^1(\Omega) \times L^2(\Omega, W_{\text{per}}(Y_1)) \times H_0^1(\Omega) \times L^2(\Omega, W_{\text{per}}(Y_2)).$$

Theorem 4.1 *The unique solution $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ of the variational problem (2.4) converges, in the sense of (4.2), to the unique solution $(u_1, \widehat{u}_1, u_2, \overline{u}_2) \in \mathcal{W}$ of the following unfolded limit problem:*

$$\begin{aligned} &\int_{\Omega \times Y_1} A(y)(\nabla u_1 + \nabla_y \widehat{u}_1)(\nabla \varphi_1 + \nabla_y \Phi_1)\, \mathrm{d}x\, \mathrm{d}y + \\ &\int_{\Omega \times Y_2} A(y)(\nabla u_2 + \nabla_y \overline{u}_2)(\nabla \varphi_2 + \nabla_y \Phi_2)\, \mathrm{d}x\, \mathrm{d}y + \\ |\Gamma| \mathcal{M}_\Gamma(h) &\int_\Omega (u_1 - u_2)(\varphi_1 - \varphi_2)\, \mathrm{d}x = \int_{\Omega \times Y_1} f(x) \varphi_1(x)\, \mathrm{d}x\, \mathrm{d}y + \\ &\int_{\Omega \times Y_2} f(x) \varphi_2(x)\, \mathrm{d}x\, \mathrm{d}y + |\Gamma| \mathcal{M}_\Gamma(g) \int_\Omega \varphi_1(x)\, \mathrm{d}x, \end{aligned} \tag{4.3}$$

for all $\varphi_1 \in H_0^1(\Omega)$, $\varphi_2 \in H_0^1(\Omega)$, $\Phi_1 \in L^2(\Omega, H^1_{per}(Y_1))$ *and* $\Phi_2 \in L^2(\Omega, H^1_{per}(Y_2))$.

Proof. Exactly like in the proof of Theorem 3.1, we unfold the variational formulation (2.4) and we take, in this case, for $\alpha \in \{1,2\}$, the admissible test functions

$$v_\alpha = \varphi_\alpha(x) + \varepsilon\omega_\alpha(x)\psi_\alpha\left(\frac{x}{\varepsilon}\right), \tag{4.4}$$

with $\varphi_\alpha, \omega_\alpha \in \mathcal{D}(\Omega)$, and $\psi_\alpha \in H^1_{\text{per}}(Y_\alpha)$. We obviously have the convergences:

$$\mathcal{T}^\varepsilon_\alpha(v_\alpha) \to \varphi_\alpha \quad \text{strongly in } L^2(\Omega \times Y_\alpha), \tag{4.5}$$

$$\mathcal{T}^\varepsilon_\alpha(\nabla v_\alpha) \to \nabla\varphi_\alpha + \nabla_y \Phi_\alpha \quad \text{strongly in } L^2(\Omega \times Y_\alpha), \tag{4.6}$$

where $\Phi_\alpha(x,y) = \omega_\alpha(x)\psi_\alpha(y)$. Then, the passage to the limit with $\varepsilon \to 0$ is standard, by using convergences (4.2) and (4.5)-(4.6). By the density of $\mathcal{D}(\Omega) \otimes H^1_{\text{per}}(Y_1)$ in $L^2(\Omega, H^1_{\text{per}}(Y_1))$ and of $\mathcal{D}(\Omega) \otimes H^1_{\text{per}}(Y_2)$ in $L^2(\Omega, H^1_{\text{per}}(Y_2))$, we are led to (4.3).

The existence and the uniqueness for the solution of problem (4.3) is a consequence of the Lax-Milgram theorem applied in the space $\mathcal{W}$ endowed with the norm $||\nabla v_1 + \nabla_y \widehat{v}_1||_{L^2(\Omega\times Y_1)} + ||\nabla v_2 + \nabla_y \overline{v}_2||_{L^2(\Omega\times Y_2)}$. Due to the uniqueness of $(u_1, \widehat{u}_1, u_2, \overline{u}_2) \in \mathcal{W}$, all the above convergences hold true for the whole sequence and this ends the proof of Theorem 4.1. ■

Theorem 4.2 *The pair* $(u_1,u_2) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *defined in Theorem* 4.1 *is the unique solution of the following homogenized system:*

$$\begin{cases} -div(A_1^{hom}\nabla u_1) + |\Gamma|\mathcal{M}_\Gamma(h)(u_1 - u_2) = |Y_1| f + |\Gamma|\mathcal{M}_\Gamma(g) & in\ \Omega, \\ -div(A_2^{hom}\nabla u_2) - |\Gamma|\mathcal{M}_\Gamma(h)(u_1 - u_2) = |Y_2| f & in\ \Omega, \\ u_1(x) = u_2(x) = 0 & on\ \partial\Omega. \end{cases} \tag{4.7}$$

The homogenized positive definite matrices A_1^{hom} *and* A_2^{hom} *are given, for* $i,j = 1,\dots,N$*, by*

$$A^{hom}_{1,ij} = \int_{Y_1} \left(a_{ij} - \sum_{k=1}^N a_{ik} \frac{\partial \chi_1^j}{\partial y_k} \right) \mathrm{d}y, \tag{4.8}$$

$$A^{hom}_{2,ij} = \int_{Y_2} \left(a_{ij} - \sum_{k=1}^N a_{ik} \frac{\partial \chi_2^j}{\partial y_k} \right) \mathrm{d}y. \tag{4.9}$$

The j*-th components of the vectorial functions* χ_1 *and* χ_2*,* $\chi_1^j \in H^1_{per}(Y_1)$ *and* $\chi_2^j \in H^1_{per}(Y_2)$ *(*$j = 1,\dots,N$*), are the unique weak solutions of the following cell problems:*

$$\begin{cases} -div_y(A(y)(\nabla_y \chi_1^j - e_j)) = 0, & in\ Y_1, \\ (A(y)(\nabla_y \chi_1^j - e_j)) \cdot n = 0, & on\ \Gamma, \\ \mathcal{M}_\Gamma(\chi_1^j) = 0, \end{cases} \tag{4.10}$$

$$\begin{cases} -div_y(A(y)(\nabla_y\chi_2^j - e_j)) = 0, & in\, Y_2, \\ (A(y)(\nabla_y\chi_2^j - e_j))\cdot n = 0, & on\, \Gamma, \\ \mathcal{M}_\Gamma(\chi_2^j) = 0, \end{cases} \tag{4.11}$$

n being the unit normal interior to Y_1. The correctors $\widehat{u}_1$ and $\overline{u}_2$ are given by

$$\widehat{u}_1(x,y) = -\sum_{j=1}^N \frac{\partial u_1}{\partial x_j}(x)\chi_1^j(y) \quad in\, \Omega\times Y_1, \tag{4.12}$$

$$\overline{u}_2(x,y) = -\sum_{j=1}^N \frac{\partial u_2}{\partial x_j}(x)\chi_2^j(y) \quad in\, \Omega\times Y_2. \tag{4.13}$$

Proof. The proof is rather standard. By choosing suitable test functions in the unfolded limit problem (4.3), we obtain:

$$\int_{\Omega\times Y_1} A(y)(\nabla u_1 + \nabla_y\widehat{u}_1)\nabla_y\Phi_1\,\mathrm{d}x\,\mathrm{d}y = 0, \quad \forall\Phi_1\in L^2(\Omega, H^1_{\text{per}}(Y_1)), \tag{4.14}$$

$$\int_{\Omega\times Y_2} A(y)(\nabla u_2 + \nabla_y\overline{u}_2)\nabla_y\Phi_2\,\mathrm{d}x\,\mathrm{d}y = 0, \quad \forall\Phi_2\in L^2(\Omega, H^1_{\text{per}}(Y_2)). \tag{4.15}$$

from which we get (4.12)-(4.13). Then, classical results from the theory of homogenization imply (4.12) and (4.10). We choose now $\Phi_1 = \Phi_2 = 0$ and $\varphi_1,\varphi_2 \in \mathcal{D}(\Omega)$ in (4.3) and we obtain

$$\int_{\Omega\times Y_1} A(y)(\nabla u_1 + \nabla_y\widehat{u}_1)\nabla\varphi_1\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega\times Y_2} A(y)(\nabla u_2 + \nabla_y\overline{u}_2)\nabla\varphi_2\,\mathrm{d}x\,\mathrm{d}y +$$
$$|\Gamma|\mathcal{M}_\Gamma(h)\int_\Omega (u_1-u_2)(\varphi_1-\varphi_2)\,\mathrm{d}x = \int_{\Omega\times Y_1} f(x)\varphi_1(x)\,\mathrm{d}x\,\mathrm{d}y +$$
$$\int_{\Omega\times Y_2} f(x)\varphi_2(x)\,\mathrm{d}x\,\mathrm{d}y + |\Gamma|\mathcal{M}_\Gamma(g)\int_\Omega \varphi_1(x)\,\mathrm{d}x, \quad \forall\varphi_1,\varphi_2\in\mathcal{D}(\Omega).$$

Taking $\varphi_2 = 0$ and using (4.12) and (4.10), we get the first equation in (4.7). In a similar way, we first get (4.11)-(4.13) and, then, the second equation in (4.7) . ∎

Remark 4.3 *The limit problem* (4.7) *is a system of two partial differential equations, fully coupled through an exchange term, namely $|\Gamma|\mathcal{M}_\Gamma(h)(u_1-u_2)$. Indeed, we notice that, in contrast to the previous case, there is no explicit relation between the two limit functions u_1 and u_2.*

Remark 4.4 *The main difference between systems* (3.17) *and* (4.7) *is the presence of a diffusion term in the second equation of* (4.7)*. From a physical point of view, this is due to the occurrence, in this last case, of a direct flow from cell to cell within the domain Ω_2^ε. From a mathematical point of view, this difference comes from the* *spaces in which the test functions can be chosen in* (3.5) *and* (4.15) *(see, for details, [38]).*

5. CONCLUSIONS

A diffusion problem in a periodic composite material formed by two constituents separated by an imperfect interface was studied by using the periodic unfolding method. Two cases were analyzed, following the connectivity of the constituents. The presence of an imperfect interface, where both the temperature and the flux exhibit jumps, led us to some modified stationary Barenblatt models. The limit systems capture the influence of the jumps in the limit temperature fields and in an additional source term. The influence of the geometry is reflected in the different forms of the corresponding limit problems (3.17) and (4.7).

REFERENCES

1. P. Donato, K. H. Le Nguyen, R. Tardieu, J. Math. Sci. (N. Y.) **6**, 176, 891-927 (2011).
2. P. Donato, S. Monsurrò, Anal. Appl. **2**, 3, 247-273 (2004).
3. H. I. Ene, D. Poliševski, Z. Angew. Math. Phys. **6**, 53, 1052-1059 (2002).
4. I. Gruais, D. Polisevski, Appl. Anal. **96**, 2, 247-260 (2017).
5. K. H. Le Nguyen, J. Elliptic Parabol. Equ. **1**, 175-188 (2015).
6. S. Monsurrò, Adv. Math. Sci. Appl. **13**, 1, 43-63 (2003).
7. D. Polisevski, R. Schiltz-Bunoiu, A. Stanescu, Bull. Math. Soc. Sci. Math. Roumanie **58** (106), 4, 463-473 (2015).
8. D. Polisevski, R. Schiltz-Bunoiu, Bulletin Scientifique, Mathématiques et Informatique **10**, 99-106 (2004).
9. C. Timofte, Comput. Math. Appl. **9**, 66, 1573-1580 (2013).
10. C. Timofte, Rom. J. Phys. **58**, 9-10, 1418-1427 (2013).
11. M. Amar, D. Andreucci, R. Gianni, Math. Model. Methods Appl. Sci. **14**, 1261-1295 (2004).
12. M. Amar, D. Andreucci, P. Bisegna, R. Gianni, Math. Meth. Appl. Sci. **29**, 767-787 (2006).
13. C. Conca, J. I. Díaz, A. Liñán, C. Timofte, Electron. J. Differential Equations **40**, 1-22 (2004).
14. J. I. Díaz, D. Gómez-Castro, C. Timofte, J. Elliptic Parabol. Equ. **2**, 117-127 (2016).
15. H. I. Ene, C. Timofte, Rom. Rep. Phys. **68**, 3, 979-989 (2016).
16. M. A. Peter, M. Böhm, Math. Methods Appl. Sci. **31**, 11, 1257-1282 (2008).
17. M. Amar, R. Gianni, Discrete Contin. Dyn. Syst. Ser. B, in press (2017).
18. R. Bunoiu, C. Timofte, Netw. Heterog. Media **11**, 4, 545-562 (2016).
19. R. Bunoiu, C. Timofte, Commun. Math. Sci. **15**, 3, 745-763 (2017).
20. K. Fellner, V. Kovtunenko, Appl. Anal. **95**, 12, 2661-2682 (2016).
21. M. Gahn, P. Knabner, M. Neuss-Radu, SIAM J. Appl. Math. **76**, 1819-1843 (2016).
22. E. R. Ijioma, A. Muntean, T. Ogawa, Combust. Theor. Model. **17**, 2, 185-223 (2013).
23. D. Cioranescu, P. Donato, Asymptot. Anal. **1**, 115-138 (1988).
24. D. Cioranescu, P. Donato, R. Zaki, Asymptot. Anal. **53**, 209-235 (2007).
25. D. Gómez, M. Lobo, M. E. Pérez, T. A. Shaposhnikova, Math. Meth. Appl. Sci. **38**, 2606-2629

(2015).
26. M. V. Goncharenko, *Asymptotic behavior of the third boundary-value problem in domains with fine-grained boundaries*, GAKUTO Int. Ser. Math. Sci. Appl. **9**, 203-213 (1997).
27. M. V. Goncharenko, L. A. Khil'kova, Ukr. Math. J. **67**, 9, 1349-1366 (2016).
28. E. Pérez, T. A. Shaposhnikova, M. N. Zubova, Dokl. Math. **90**, 1, 489-494 (2014).
29. A. Piatnitski, V. Rybalko, J. Math. Sci. **177**, 1, 109-140 (2011).
30. D. Cioranescu, A. Damlamian, P. Donato, G. Griso, R. Zaki, SIAM J. Math. Anal. **4**, 4, 718-760 (2012).
31. D. Cioranescu, A. Damlamian, G. Griso, SIAM J. Math. Anal. **40**, 4, 1585-1620 (2008).
32. G. I. Barenblatt, I. P. Zheltov, I. N. Kochina, J. Appl. Math. Mech. **24**, 1286-1303 (1960).
33. R. E. Showalter, *Microstructure Models of Porous Media*, Homogenization and Porous Media, Springer, 183-202 (1997).
34. G. Allaire, SIAM J. Math. Anal. **23**, 1482-1518 (1992).
35. T. Arbogast, J. Douglas Jr., U. Hornung, SIAM J. Math. Anal. **21**, 823-836 (1990).
36. M. Mabrouk, H. Samadi, Math. Methods Appl. Sci. **27**, 405-425 (2004).
37. R. E. Showalter, D. B. Visarraga, J. Math. Anal. Appl. **295**, 191-201 (2004).
38. R. Bunoiu, C. Timofte, *Upscaling of a diffusion problem with interfacial flux jump leading to a modified Barenblatt model*, Z. Angew. Math. Mech., *in press*, 2018; DOI: 10.1002/zamm.201800018.