TWO (3+1)-DIMENSIONAL GARDNER-TYPE EQUATIONS WITH MULTIPLE KINK SOLUTIONS

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Abstract. In this work we introduce two (3+1)-dimensional Gardner-type equations. We find the necessary conditions for the multiple kink solutions to exist. We show that each model possesses distinct conditions to guarantee the existence of multiple kink solutions.

Key words: (3+1)-dimensional Gardner-type equation; Hirota bilinear method; multiple kink solutions.

1. INTRODUCTION

The (1+1)-dimensional Gardner equation reads

$$u_t + 6\beta u u_x + u_{xxx} - \frac{3}{2}\alpha^2 u^2 u_x = 0, \tag{1}$$

which combines the Korteweg-de Vries (KdV) equation and the modified KdV (mKdV) equation. However, the (2+1)-dimensional Gardner equation is given in an integro-differential form as

$$u_t + 6\beta u u_x + u_{xxx} - \frac{3}{2}\alpha^2 u^2 u_x + 3\sigma^2 \partial_x^{-1} u_{yy} - 3\alpha\sigma u_x \partial_x^{-1} u_y = 0, \tag{2}$$

where $\sigma^2 = \pm 1$ and α and β are arbitrary constants, and ∂_x^{-1} is the inverse of ∂_x with $\partial_x \partial_x^{-1} = \partial_x^{-1}\partial_x = 1$. The Gardner equations (1) and (2) were examined in Refs. [1–10] by using a variety of methods, such as the inverse scattering method, the Casorati and Grammian determinant solutions, and the Hirota's direct method.

The Gardner equations (1) and (2) are widely used in various branches of physics, such as plasma physics, fluid physics, and quantum field theory [1–7].

The Lax pair [1–3] for the (2+1)-dimensional Gardner equation (2) is given by

$$\sigma\psi_y + \psi_{xx} + \alpha u \psi_x + \beta u \psi = 0, \tag{3}$$

$$\psi_t + 4\psi_{xxx} + \alpha u \psi_{xx} + (3\alpha u_x + \tfrac{3}{2}\alpha^2 u^2 + 6\beta u - 3\alpha\sigma\partial_x^{-1} u_y)\psi_x$$
$$+ (3\beta u_x + \tfrac{3}{2}\alpha^2 \beta u^2 + 3\beta\sigma\partial_x^{-1} u_y)\psi = 0. \tag{4}$$

The compatibility condition between (3) and (4) gives the (2+1)-dimensional Gardner equation (2) [1]. Many kinds of solitons occur for this equation, including

pulse-type solitons, positive and negative solitons, travelling wave solutions, kinks and table-top solitons [1].

In this work, we introduce two (3+1)-dimensional Gardner-type equations, given in an integro-differential form by

$$u_t + 6\beta u u_x + u_{xxx} - \tfrac{3}{2}\alpha^2 u^2 u_x + 3\sigma^2 \partial_x^{-1} u_{yy} - 3\alpha\sigma u_x \partial_x^{-1} u_y \\ + 3\sigma^2 \partial_x^{-1} u_{zz} = 0, \tag{5}$$

and

$$u_t + 6\beta u u_x + u_{xxx} - \tfrac{3}{2}\alpha^2 u^2 u_x + 3\sigma^2 \partial_x^{-1} u_{yy} - 3\alpha\sigma u_x \partial_x^{-1} u_y \\ + 3\sigma^2 \partial_x^{-1} u_{zz} - 3\alpha\sigma u_x \partial_x^{-1} u_z = 0, \tag{6}$$

where the terms $3\sigma^2 \partial_x^{-1} u_{zz}$ and $3\sigma^2 \partial_x^{-1} u_{zz} - 3\alpha\sigma u_x \partial_x^{-1} u_z$ are added, respectively, to the (2+1)-dimensional Gardner equation (2).

People working in the area of nonlinear partial differential equations developed many powerful methods, such as the Bäcklund transformation method, the Darboux transformation, the Pfaffian technique, the inverse scattering method, the Painlevé analysis, the generalized symmetry method and other methods [8]- [20]. The computer symbolic systems such as Maple and Mathematica allow us to perform complicated and tedious calculations.

It is the aim of this work to focus on studying the necessary conditions for multiple kink solitons to exist for the extended models (5) and (6). The constraint conditions for the existence of multiple kink solutions will be established. The simplified Hirota's method, developed by Hereman-Nuseir [13] will be employed to achieve this goal.

2. THE FIRST (3+1)-DIMENSIONAL GARDNER-TYPE EQUATION

In this Section, we will derive multiple kink solutions for the first (3+1)-dimensional Gardner-type equation

$$u_t + 6\beta u u_x + u_{xxx} - \frac{3}{2}\alpha^2 u^2 u_x + 3\sigma^2 \partial_x^{-1} u_{yy} - 3\alpha\sigma u_x \partial_x^{-1} u_y + 3\sigma^2 \partial_x^{-1} u_{zz} = 0, \tag{7}$$

where ∂_x^{-1} is the inverse of ∂_x with $\partial_x \partial_x^{-1} = \partial_x^{-1} \partial_x = 1$, and

$$(\partial_x^{-1} f)(x) = \int_{-\infty}^{x} f(t)dt, \tag{8}$$

under the decaying condition at infinity. The first (3+1)-dimensional Gardner-type equation is proposed by adding one term $3\sigma^2 \partial_x^{-1} u_{zz}$ to the (2+1)-dimensional Gardner equation (2).

To get rid of the inverse operator, we use the potential transformation

$$u(x, y, z, t) = v_x(x, y, z, t), \tag{9}$$

which carries (7) to the potential (3+1)-dimensional form

$$v_{xt} + 6\beta v_x v_{xx} + v_{xxxx} - \frac{3}{2}\alpha^2 v_x^2 v_{xx} + 3\sigma^2 v_{yy} - 3\alpha\sigma v_{xx}v_y + 3\sigma^2 v_{zz} = 0. \quad (10)$$

We first substitute

$$v(x,y,z,t) = e^{k_i x + r_i y + s_i z - c_i t}, \quad (11)$$

into the linear terms of (10) to obtain the dispersion relation c_i as

$$c_i = k_i^3 + \frac{3\sigma^2(r_i^2 + s_i^2)}{k_i}, \quad (12)$$

which gives the dispersion variable as

$$\theta_i = k_i x + r_i y + s_i z - (k_i^3 + \frac{3\sigma^2(r_i^2 + s_i^2)}{k_i})t. \quad (13)$$

We next use the Cole-Hopf transformation

$$u(x,y,z,t) = R(\ln f(x,y,z,t))_x, \quad (14)$$

which gives

$$v(x,y,z,t) = R(\ln f(x,y,z,t)). \quad (15)$$

The simplified Hirota's method admits the use of the auxiliary function $f(x,y,z,t)$ for the single kink solution by

$$f(x,y,z,t) = 1 + e^{k_1 x + r_1 y + s_1 z - (k_1^3 + \frac{3\sigma^2(r_1^2 + s_1^2)}{k_1})t}. \quad (16)$$

Substituting (15) into (10) and solving for R we find

$$R = \frac{2}{\alpha}, \quad (17)$$

and the kink solutions exist if the coefficients r_i satisfy the constraint condition

$$r_i = \frac{k_i(2\beta - \alpha k_i)}{\sigma\alpha}, \; i = 1, 2, \cdots, N, \quad (18)$$

and $s_i, i = 1, 2\cdots, N$ are left as free parameters. Based on this result, the dispersion relation (12) becomes

$$c_i = k_i^3 + \frac{3k_i(2\beta - \alpha k_i)^2}{\alpha^2} + \frac{3\sigma^2 s_i^2}{k_i}, \; i = 1, 2, \cdots, N. \quad (19)$$

Using the potential (9) gives the single-kink solution

$$u(x,y,z,t) = \frac{2k_1 e^{k_1 x + \frac{k_1(2\beta - \alpha k_1)}{\sigma\alpha}y + s_1 z - \left(k_1^3 + \frac{3k_1(2\beta - \alpha k_1)^2}{\alpha^2} + \frac{3\sigma^2 s_1^2}{k_1}\right)t}}{\alpha\left(1 + e^{k_1 x + \frac{k_1(2\beta - \alpha k_1)}{\sigma\alpha}y + s_1 z - \left(k_1^3 + \frac{3k_1(2\beta - \alpha k_1)^2}{\alpha^2} + \frac{3\sigma^2 s_1^2}{k_1}\right)t}\right)}. \quad (20)$$

For the two-kink solutions we set

$$f(x,y,z,t) = 1 + e^{\theta_1} + e^{\theta_2}. \tag{21}$$

Proceeding as before, we find the two-kink solutions as

$$u(x,y,z,t) = \frac{2\sum_{i=1}^{2} k_i\, e^{k_i x + \frac{k_i(2\beta - \alpha k_i)}{\sigma\alpha}y + s_i z - \left(k_i^3 + \frac{3k_i(2\beta - \alpha k_i)^2}{\alpha^2} + \frac{3\sigma^2 s_i^2}{k_i}\right)t}}{\alpha\left(1 + \sum_{i=1}^{2} e^{k_i x + \frac{k_i(2\beta - \alpha k_i)}{\sigma\alpha}y + s_i z - \left(k_i^3 + \frac{3k_i(2\beta - \alpha k_i)^2}{\alpha^2} + \frac{3\sigma^2 s_i^2}{k_i}\right)t}\right)}. \tag{22}$$

For the three-kink solutions, we set

$$f(x,y,z,t) = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3}. \tag{23}$$

Proceeding as before, we find the three-kink solutions given by

$$u(x,y,z,t) = \frac{2\sum_{i=1}^{3} k_i\, e^{k_i x + \frac{k_i(2\beta - \alpha k_i)}{\sigma\alpha}y + s_i z - \left(k_i^3 + \frac{3k_i(2\beta - \alpha k_i)^2}{\alpha^2} + \frac{3\sigma^2 s_i^2}{k_i}\right)t}}{\alpha\left(1 + \sum_{i=1}^{3} e^{k_i x + \frac{k_i(2\beta - \alpha k_i)}{\sigma\alpha}y + s_i z - \left(k_i^3 + \frac{3k_i(2\beta - \alpha k_i)^2}{\alpha^2} + \frac{3\sigma^2 s_i^2}{k_i}\right)t}\right)}. \tag{24}$$

This shows that the (3+1)-dimensional Gardner–type equation (7) has N-kink solutions for finite N, where $N \geq 1$.

3. THE SECOND (3+1)-DIMENSIONAL GARDNER-TYPE EQUATION

In this Section, we will derive multiple kink solutions for the second (3+1)-dimensional Gardner-type equation

$$\begin{aligned} u_t + 6\beta u u_x + u_{xxx} - \tfrac{3}{2}\alpha^2 u^2 u_x + 3\sigma^2 \partial_x^{-1} u_{yy} - 3\alpha\sigma u_x \partial_x^{-1} u_y \\ + 3\sigma^2 \partial_x^{-1} u_{zz} - 3\alpha\sigma u_x \partial_x^{-1} u_z = 0, \end{aligned} \tag{25}$$

where we added the two terms $3\sigma^2 \partial_x^{-1} u_{zz}$ and $-3\alpha\sigma u_x \partial_x^{-1} u_z$ to the (2+1)-dimensional Gardner equation (2).

The potential transformation

$$u(x,y,z,t) = v_x(x,y,z,t), \tag{26}$$

carries (25) to the potential (3+1)-dimensional form

$$v_{xt} + 6\beta v_x v_{xx} + v_{xxxx} - \frac{3}{2}\alpha^2 v_x^2 v_{xx} + 3\sigma^2 v_{yy} - 3\alpha\sigma v_{xx} v_y + 3\sigma^2 v_{zz} - 3\alpha\sigma v_{xx} v_z = 0. \tag{27}$$

Substituting

$$v(x,y,z,t) = e^{k_i x + r_i s_i z - c_i t}, \tag{28}$$

into the linear terms of (27) gives the dispersion relation c_i as

$$c_i = k_i^3 + \frac{3\sigma^2(r_i^2 + s_i^2)}{k_i}, \tag{29}$$

which gives the dispersion variable as

$$\theta_i = k_i x + r_i y + s_i z - (k_i^3 + \frac{3\sigma^2(r_i^2 + s_i^2)}{k_i})t. \tag{30}$$

We next use the Cole-Hopf transformation

$$u(x,y,z,t) = R\left(\ln f(x,y,z,t)\right)_x, \tag{31}$$

or equivalently

$$v(x,y,z,t) = R\left(\ln f(x,y,z,t)\right). \tag{32}$$

The simplified Hirota's method admits the use of the auxiliary function $f(x,y,z,t)$ for the single kink solution by

$$f(x,y,z,t) = 1 + e^{k_1 x + r_1 y + s_1 z - (k_1^3 + \frac{3\sigma^2(r_1^2 + s_1^2)}{k_1})t}. \tag{33}$$

Substituting (32) into (27) and solving for R we find

$$R = \frac{2}{\alpha}, \tag{34}$$

and the kink solutions exist if the coefficients r_i satisfy the constraint condition

$$r_i = \frac{(2k_i\beta - \alpha k_i^2 - \sigma\alpha s_i)}{\sigma\alpha}, \; i = 1, 2, \cdots, N, \tag{35}$$

and $s_i, i = 1, 2\cdots, N$ are left as free parameters. Based on this result, the dispersion relation (29) becomes

$$c_i = k_i^3 + \frac{3k_i(2k_i\beta - \alpha k_i^2 - \sigma\alpha s_i)^2}{\alpha^2} + \frac{3\sigma^2 s_i^2}{k_i}, \; i = 1, 2, \cdots, N. \tag{36}$$

Using the potential (26) gives the single-kink solution

$$u(x,y,z,t) = \frac{2k_1 e^{k_1 x + \frac{(2k_1\beta - \alpha k_1^2 - \sigma\alpha s_1)}{\sigma\alpha}y + s_1 z - c_i t}}{\alpha\left(1 + e^{k_1 x + \frac{(2k_1\beta - \alpha k_1^2 - \sigma\alpha s_1)}{\sigma\alpha}y + s_1 z - c_i t}\right)}, \tag{37}$$

where c_i is given in (36).

For the two-kink solutions we set

$$f(x,y,z,t) = 1 + e^{\theta_1} + e^{\theta_2}, \tag{38}$$

and we find that the two-kink solutions, and hence the three-kink solutions exist only if

$$s_i = \gamma k_i, i = 1, 2, \cdots, N, \tag{39}$$

where γ is a real number. This in turn gives the two-kink solutions as

$$u(x,y,z,t) = \frac{2\sum_{i=1}^{2} k_i \, e^{k_i x + \frac{(2k_i\beta - \alpha k_i^2 - \sigma\alpha s_i)}{\sigma\alpha}y + s_i z - c_i t}}{\alpha\left(1 + \sum_{i=1}^{2} e^{k_i x + \frac{(2k_i\beta - \alpha k_i^2 - \sigma\alpha s_i)}{\sigma\alpha}y + s_i z - c_i t}\right)}, \qquad (40)$$

and the three-kink solutions as

$$u(x,y,z,t) = \frac{2\sum_{i=1}^{3} k_i \, e^{k_i x + \frac{(2k_i\beta - \alpha k_i^2 - \sigma\alpha s_i)}{\sigma\alpha}y + s_i z - c_i t}}{\alpha\left(1 + \sum_{i=1}^{3} e^{k_i x + \frac{(2k_i\beta - \alpha k_i^2 - \sigma\alpha s_i)}{\sigma\alpha}y + s_i z - c_i t}\right)}, \qquad (41)$$

where c_i is given in (36).

4. DISCUSSION

We introduced two (3+1)-dimensional Gardner-type equations. We found the necessary conditions for multiple kink solutions to exist. The dispersion relations and the corresponding constraint conditions were also derived.

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