A COMPUTATIONALLY EFFICIENT METHOD FOR A CLASS OF FRACTIONAL VARIATIONAL AND OPTIMAL CONTROL PROBLEMS USING FRACTIONAL GEGENBAUER FUNCTIONS

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This paper is devoted to investigate, from the numerical point of view, fractional-order Gegenbauer functions to solve fractional variational problems and fractional optimal control problems. We first introduce an orthonormal system of fractional-order Gegenbauer functions. Then, a formulation for the fractional-order Gegenbauer operational matrix of fractional integration is constructed. An error upper bound for the operational matrix of the fractional integration is also given. The properties of the fractional-order Gegenbauer functions are utilized to reduce the given optimization problems to systems of algebraic equations. Some numerical examples are included to demonstrate the efficiency and the accuracy of the proposed approach.

Key words: Fractional variational problems; Fractional optimal control problems; Fractional-order Gegenbauer functions.

1. INTRODUCTION

There is an increasing interest in the study of dynamic systems of fractional order. Extending derivatives and integrals from integer to non-integer order has a firm and long standing theoretical foundation. Fractional differentiation is nowadays an area of current strong research with many different and important applications in different field of sciences including physics, signal processing, fluid mechanics, viscoelasticity, mathematical biology, electrochemistry, chemistry, economics, engineering, and control theory [1–6].

The calculus of variations has communication with some branches of sciences and engineering, such as differential equations, geometry, control theory, economics, and electrical engineering [7–10]. The classical variational calculus is extended to the fractional variational calculus. Agrawal [11] and Klimek [12] were among the earliest researchers who developed formulations and achieved the necessary optimality conditions for various types of fractional variational problems (FVPs) with respect to Riemann-Liouville and Caputo fractional derivatives. Pooseh *et al.* [13] used discrete time methods for getting the solution of FVPs, while Almeida *et al.* [14] introduced numerical solutions of FVPs depending on indefinite integrals. Also, some numerical methods have been constructed for solving isoperimetric FVPs [15, 16] and multiobjective FVPs [17].

Optimal control theory is an area in mathematics that has been under development for years but the fractional optimal control theory is a very new area in mathematics. The optimal control problem refers to the minimization of a performance index subject to dynamic constraints on the state and control variables. A fractional optimal control problem (FOCP) is an optimal control problem in which either the performance index or the differential equations governing the dynamic of the system or both contain at least one fractional-order derivative term. In recent years, FOCPs have gained much attention for their many applications in engineering and physics. Many researchers have been interested in studying FOCPs and finding numerical solutions for them, see, for instance [18–20]. Agrawal [21] provides Hamiltonian formulas for FOCPs with Caputo fractional derivatives. Also, Frederico and Torres [22] formulated a Noether-type theorem in the general context of the fractional optimal control in the sense of Caputo fractional derivatives. Lotfi et al. [23] introduced a numerical technique by using the operational matrices of fractional Riemann-Liouville integration and multiplication, together with the Lagrange multiplier method for solving the FOCP with a quadratic performance index. Recently, several fractional integration/differentiation matrices have been developed for numerical treatment of fractional differential equations, FOCPs and FVPs see *e.g.* [24–29].

In the current paper, we investigate and develop a new numerical technique to solve some types of FOCPs and FVPs described as follows

1. The fractional variational problem:

$$\text{Minimize } J = \int_0^1 F\big(t, x(t), D^\nu x(t)\big)\,dt, \qquad 0 < \nu \le 1, \tag{1}$$

   subjected to the initial condition,

$$x(0) = x_0. \tag{2}$$

2. The fractional optimal control problem:

$$\text{Minimize } J[x,u] = \int_0^1 F\big(t, x(t), u(t)\big)dt, \tag{3}$$

subjected to the dynamic constraint,

$$D^{\nu}x(t) = g\big(t, x(t)\big) + b(t)u(t), \qquad 0 < \nu \leq 1, b(t) \neq 0, \tag{4}$$

and the initial condition,

$$x(0) = x_0. \tag{5}$$

The main goal of this paper is to introduce a numerical technique to obtain an efficient numerical solutions for the FVP (1)-(2) and the FOCP (3)-(5). Our technique is depending on the fractional-order Gegenbauer functions (FGFs) and Rayleigh-Ritz method. In order to solve the FVP, the fractional derivative of the function $x(t)$ is expanding by means of the FGFs with unknown coefficients using the operational matrix of fractional integrals. Then the FVP (1)-(2) can be reduced to a problem consisting of solving a system of algebraic equations by using the Rayleigh-Ritz method. In the case of the FOCP, we extract the control variable $u(t)$ from the dynamic constraint (4) and use that in the performance index to turn the FOCP into FVP and then following the same technique.

The remainder of this paper is organized as follows: In Sec. 2, we introduce some preliminaries and fundamentals of fractional calculus with some properties of shifted Gegenbauer polynomials. In Sec. 3, we define the FGFs with its properties and then we derive its operational matrix of fractional integrals with its error bound. In Sec. 4 and Sec. 5, our numerical approaches for solving the FVPs and FOCPs are introduced. In Sec. 6, several numerical examples are introduced with comparisons between our results and those achieved using other numerical methods. Finally, in Sec. 7 presents the main conclusion.

2. PRELIMINARIES

In this Section, we present the definitions and mathematical tools. We first recall the definitions of fractional integrals and derivatives. We then summarize some important properties of Gegenbauer polynomials.

2.1. SOME PRELIMINARIES OF FRACTIONAL CALCULUS

Definition 2.1. The left-sided fractional integral of order $\gamma > 0$ is defined by

$$\begin{aligned} I^{\gamma}f(t) &= \frac{1}{\Gamma(\gamma)}\int_0^t (t-y)^{\gamma-1}f(y)dy, \quad \gamma > 0, \quad t > 0, \\ I^0 f(t) &= f(t), \end{aligned} \tag{6}$$

where

$$\Gamma(\gamma) = \int_0^{\infty} t^{\gamma-1} e^{-t} dt.$$

The integral operator I^{γ} satisfies:

$$\begin{aligned} &I^{\gamma} I^{\delta} f(t) = I^{\gamma+\delta} f(t), \\ &I^{\gamma} I^{\delta} f(t) = I^{\delta} I^{\gamma} f(t), \\ &I^{\gamma} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\gamma)}\, t^{\beta+\gamma}. \end{aligned} \tag{7}$$

Definition 2.2. The left-sided Caputo fractional derivative of order γ is defined by

$$D^{\gamma} f(t) = \frac{1}{\Gamma(\lceil\gamma\rceil - \gamma)} \int_0^t (t-y)^{\lceil\gamma\rceil-\gamma-1} \frac{d^{\lceil\gamma\rceil}}{dy^{\lceil\gamma\rceil}} f(y) dy, \qquad t > 0. \tag{8}$$

The operator D^{γ} satisfies:

$$\begin{aligned} &D^{\gamma} C = 0, \qquad (C \text{ is constant}), \\ &I^{\gamma} D^{\gamma} h(t) = h(t) - \sum_{i=0}^{\lceil\gamma\rceil-1} h^{(i)}(0^{+}) \frac{t^i}{i!}, \\ &D^{\gamma} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\gamma)}\, t^{\beta-\gamma}, \\ &D^{\gamma}(\lambda g(t) + \mu h(t)) = \lambda D^{\gamma} g(t) + \mu D^{\gamma} h(t). \end{aligned} \tag{9}$$

2.2. GEGENBAUER POLYNOMIALS

We denote $G_j^{(\alpha)}(z);\ \alpha > -\frac{1}{2}$ as the j-th order Gegenbauer polynomial defined on the interval $[-1,1]$. The Gegenbauer polynomials standardized by Doha [30] are given by

$$G_j^{(\alpha)}(z) = \frac{\Gamma(2\alpha+j)\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+j+\frac{1}{2})\Gamma(2\alpha)} P_j^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(z), \tag{10}$$

where $P_j^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(z)$ is the Jacobi polynomial of degree j and associated with the parameters $\alpha-\frac{1}{2},\ \alpha-\frac{1}{2}$.

All classic orthogonal polynomials, $G_j^{(\alpha)}(z)$ constitute an orthogonal system with respect to the weight function $\omega^{(\alpha)}(z) = (1-z^2)^{\alpha-\frac{1}{2}}$, *i.e.*,

$$\int_{-1}^{1} G_j^{(\alpha)}(z) G_k^{(\alpha)}(z) w^{(\alpha)}(z) dz = \delta_{jk} h_k^{(\alpha)}, \tag{11}$$

where δ_{jk} is the Kronecker function and

$$h_k^{(\alpha)} = \frac{2^{1-2\alpha}\pi\Gamma(2\alpha+k)}{(\alpha+k)k!\Gamma^2(\alpha)}.$$

For using these polynomials on $[0,L]$, we present the so called shifted Gegenbauer polynomials by implementing the change of variable $z=(\frac{2x}{L}-1)$. Then the shifted Gegenbauer polynomials $G^{(\alpha)}_{L,j}(x)$; $G^{(\alpha)}_{L,j}(x)\equiv G^{(\alpha)}_{j}(\frac{2x}{L}-1)$ are constituting an orthogonal system with respect to the weight function $\omega^{(\alpha)}_{L}(x)=(Lx-x^2)^{\alpha-\frac{1}{2}}$ in the interval $[0,L]$ with the orthogonality property:

$$\int_0^L G^{(\alpha)}_{L,j}(x)G^{(\alpha)}_{L,k}(x)w^{(\alpha)}_{L}(x)dt=h^{(\alpha)}_{L,k}\delta_{jk}, \tag{12}$$

where

$$h^{(\alpha)}_{L,k}=(\frac{L}{2})^{2\alpha}h^{(\alpha)}_{k}=\frac{2^{1-4\alpha}\pi\Gamma(2\alpha+k)L^{2\alpha}}{(\alpha+k)k!\Gamma^2(\alpha)}.$$

The shifted Gegenbauer polynomials are generated from the three-term recurrence relations

$$G^{(\alpha)}_{L,j}(x)=\frac{(\alpha+j-1)(2x-L)}{L}G^{(\alpha)}_{L,j-1}(x)-(2\alpha+j-2)G^{(\alpha)}_{L,j-2}(x), \tag{13}$$

with

$$\begin{aligned} G^{(\alpha)}_{L,0}(x)&=1,\\ G^{(\alpha)}_{L,1}(x)&=\frac{2\alpha(2x-L)}{L}. \end{aligned} \tag{14}$$

The explicit analytic form of $G^{(\alpha)}_{L,j}(x)$ is given by

$$G^{(\alpha)}_{L,j}(x)=\sum_{k=0}^{j}\frac{(-1)^{j-k}\Gamma(2\alpha+j+k)\Gamma(\alpha+\frac{1}{2})}{\Gamma(2\alpha)\Gamma(\alpha+k+\frac{1}{2})(j-k)!k!L^k}x^k. \tag{15}$$

The endpoint values of the shifted Gegenbauer polynomial are given as

$$\begin{aligned} G^{(\alpha)}_{L,j}(0)&=(-1)^j\frac{\Gamma(2\alpha+j)}{\Gamma(2\alpha)j!},\\ G^{(\alpha)}_{L,j}(L)&=\frac{\Gamma(2\alpha+j)}{\Gamma(2\alpha)j!}. \end{aligned} \tag{16}$$

Let

$$\mathbb{G}^{(\alpha)}_{L,N}=\text{Span}\left\{G^{(\alpha)}_{L,0}(x),G^{(\alpha)}_{L,1}(x),\cdots,G^{(\alpha)}_{L,N}(x)\right\},$$

and y be an arbitrary element in $L^2_{w^{(\alpha)}}[0,L]$. Since $\mathbb{G}^{(\alpha)}_{L,N}$ is a finite dimensional vector space, y has the unique best approximation out of $\mathbb{G}^{(\alpha)}_{L,N}$ like $y_N\in\mathbb{G}^{(\alpha)}_{L,N}$, such that

$$\forall g\in\mathbb{G}^{(\alpha)}_{L,N},\qquad \|y-y_N\|_2\leq\|y-g\|_2,$$

where $\|y\|_2=\sqrt{\langle y,y\rangle_{w^{(\alpha)}}}$.

Any square integrable function $y(x)$ defined on the interval $[0,L]$, may be expressed in terms of shifted Gegenbauer polynomials as

$$y(x) = \sum_{j=0}^{\infty} y_j G_{L,j}^{(\alpha)}(x),$$

from which the coefficients y_j are given by

$$y_j = \frac{1}{h_{L,j}^{(\alpha)}} \int_0^L y(x) G_{L,j}^{(\alpha)}(x) w_L^{(\alpha)}(x) dt, \qquad j = 0,1,\ldots. \tag{17}$$

If we approximate $y(x)$ by the first $(N+1)$-terms, then we can write

$$y_N(x) \simeq \sum_{j=0}^{N} y_j G_{L,j}^{(\alpha)}(x), \tag{18}$$

which alternatively may be written in the matrix form:

$$y_N(x) \simeq \mathbf{Y}^T \Theta_{L,N}(x), \tag{19}$$

with

$$\mathbf{Y}^T = [y_0, y_1, \cdots, y_N], \tag{20}$$

and

$$\Theta_{L,N}(x) = [G_{L,0}^{(\alpha)}(x), G_{L,1}^{(\alpha)}(x), \cdots, G_{L,N}^{(\alpha)}(x)]^T. \tag{21}$$

3. FRACTIONAL GEGENBAUER FUNCTIONS

Now, we introduce some properties of fractional orthogonal functions based on shifted Gegenbauer polynomials to obtain the solution of FVPs and FOCPs more simply and efficiently. The FGFs can be defined on the interval $t \in [0,1]$ by introducing the change of variable $x = t^\mu$ and $\mu > 0$ on shifted Gegenbauer polynomials. Let the FGFs $G_{1,i}^{(\alpha)}(t^\mu)$ be denoted by $G_i^{(\alpha,\mu)}(t)$. The explicit analytic form is given as

$$G_i^{(\alpha,\mu)}(t) = G_{1,i}^{(\alpha)}(t^\mu) = \sum_{k=0}^{i} \frac{(-1)^{i-k}\Gamma(2\alpha+i+k)\Gamma(\alpha+\frac{1}{2})}{\Gamma(2\alpha)\Gamma(\alpha+k+\frac{1}{2})(i-k)!k!} t^{\mu k}, \tag{22}$$

that implies

$$\begin{aligned} G_i^{(\alpha,\mu)}(0) &= (-1)^i \frac{\Gamma(2\alpha+i)}{\Gamma(2\alpha)i!}, \\ G_i^{(\alpha,\mu)}(1) &= \frac{\Gamma(2\alpha+i)}{\Gamma(2\alpha)i!}, \end{aligned} \tag{23}$$

and we have

$$\int_0^1 G_j^{(\alpha,\mu)}(t)G_k^{(\alpha,\mu)}(t)w^{(\alpha,\mu)}(t)dt = h_{1,k}^{(\alpha)}\delta_{jk}, \tag{24}$$

where $w^{(\alpha,\mu)}(t) = \mu\, t^{\mu(\alpha+\frac{1}{2})-1}(1-t^{\mu})^{\alpha-\frac{1}{2}}$, thanks to (11),

Remark 3.1 *The FGFs comprise unlimited number of orthogonal functions, among them, the shifted fractional-order Chebyshev functions of the first kind $T_i^{(\mu)}(t)$, the shifted fractional-order Chebyshev functions of the second kind $U_i^{(\mu)}(t)$, and the shifted fractional-order Legendre functions $L_i^{(\mu)}(t)$. These orthogonal functions are interrelated to the FGFs by the following relations (see [31,32])*

$$\begin{aligned} T_i^{(\mu)}(t) &= G_i^{(0,\mu)}(t),\\ U_i^{(\mu)}(t) &= (i+1)G_i^{(1,\mu)}(t),\\ L_i^{(\mu)}(t) &= G_i^{(\frac{1}{2},\mu)}(t). \end{aligned} \tag{25}$$

Similarly, a function $y(t)$ defined over the interval $[0,1]$ can be expanded in terms of FGFs as

$$y(t) \simeq y_N(t) \equiv \sum_{j=0}^{N} c_j G_j^{(\alpha,\mu)}(t) = C^T \Psi_N(t), \tag{26}$$

where

$$C^T \equiv [c_0, c_1, \cdots, c_N], \qquad \Psi_N(t) \equiv [G_0^{(\alpha,\mu)}(t), G_1^{(\alpha,\mu)}(t), \cdots, G_N^{(\alpha,\mu)}(t)]^T, \tag{27}$$

and

$$c_j = \frac{1}{h_{1,j}^{(\alpha)}} \int_0^1 w^{(\alpha,\mu)}(t)y(t)G_j^{(\alpha,\mu)}(t)dt, \qquad j = 0, 1, \cdots. \tag{28}$$

Theorem 3.2 *Let $G_N^{(\alpha,\mu)}(t)$ be the shifted fractional-order Gegenbauer vector and let also $0 < \nu \leq 1$. Then the Riemann-Liouville fractional integral of order ν of $\Psi_N(t)$ is given by*

$$I^{\nu}\Psi_N(t) = \boldsymbol{I}^{(\nu)}\Psi_N(t), \tag{29}$$

where $\boldsymbol{I}^{(\nu)}$ is the $(N+1)\times(N+1)$ operational matrix of fractional integral of order

ν and is defined as follows:

$$\boldsymbol{I}^{(\nu)}=\begin{pmatrix}\Delta_\nu(0,0,\alpha,\mu) & \Delta_\nu(0,1,\alpha,\mu) & \cdots & \Delta_\nu(0,j,\alpha,\mu) & \cdots & \Delta_\nu(0,N,\alpha,\mu)\\ \Delta_\nu(1,0,\alpha,\mu) & \Delta_\nu(1,1,\alpha,\mu) & \cdots & \Delta_\nu(1,j,\alpha,\mu) & \cdots & \Delta_\nu(1,N,\alpha,\mu)\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ \Delta_\nu(i,0,\alpha,\mu) & \Delta_\nu(i,1,\alpha,\mu) & \cdots & \Delta_\nu(i,j,\alpha,\mu) & \cdots & \Delta_\nu(i,N,\alpha,\mu)\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ \Delta_\nu(N,0,\alpha,\mu) & \Delta_\nu(N,1,\alpha,\mu) & \cdots & \Delta_\nu(N,j,\alpha,\mu) & \cdots & \Delta_\nu(N,N,\alpha,\mu)\end{pmatrix}, \tag{30}$$

where

$$\Delta_\nu(i,j,\alpha,\mu)=\sum_{k=0}^{i}\frac{(-1)^{i-k}\,2\,\Gamma(2\alpha+i+k)\Gamma(\alpha+\frac{1}{2})\Gamma(\mu k+1)}{\Gamma(\alpha+k+\frac{1}{2})\Gamma(\mu k+\nu+1)(i-k)!k!}$$
$$\times\sum_{s=0}^{j}\frac{(-1)^{j-s}(\alpha+j)\,j!\,\Gamma(2\alpha+j+s)\Gamma(k+s+\alpha+\frac{\nu}{\mu}+\frac{1}{2})}{\Gamma(2\alpha+j)\Gamma(\alpha+s+\frac{1}{2})\Gamma(k+s+2\alpha+\frac{\nu}{\mu}+1)(j-s)!s!}. \tag{31}$$

Proof. Using Eqs. (9) and (22), the fractional integral of order μ for the FGFs $G_i^{(\alpha,\mu)}(t)$ is given by

$$\begin{aligned}I^\nu G_i^{(\alpha,\mu)}(t)&=\sum_{k=0}^{i}\frac{(-1)^{i-k}\Gamma(2\alpha+i+k)\Gamma(\alpha+\frac{1}{2})}{\Gamma(2\alpha)\Gamma(\alpha+k+\frac{1}{2})(i-k)!k!}D^\nu t^{\mu k}\\&=\sum_{k=0}^{i}\frac{(-1)^{i-k}\Gamma(2\alpha+i+k)\Gamma(\alpha+\frac{1}{2})\Gamma(\mu k+1)}{\Gamma(2\alpha)\Gamma(\alpha+k+\frac{1}{2})\Gamma(\mu k+\nu+1)(i-k)!\,k!}t^{\mu k+\nu}.\end{aligned} \tag{32}$$

Now, approximating $t^{\mu k-\nu}$ by the first $(N+1)$ terms of FGFs $G_j^{(\alpha,\mu)}(t)$, we have

$$t^{\mu k+\nu}\simeq\sum_{j=0}^{N}\eta_{k,j}G_j^{(\alpha,\mu)}(t), \tag{33}$$

where $\eta_{k,j}$ is given from (27) with $y(t)=t^{\mu k+\nu}$, and this immediately gives

$$\eta_{k,j}=\sum_{s=0}^{j}\frac{(-1)^{j-s}(\alpha+j)j!\Gamma^2(\alpha)\Gamma(2\alpha+j+s)\Gamma^2(\alpha+\frac{1}{2})\Gamma(k+s+\alpha+\frac{\nu}{\mu}+\frac{1}{2})\Gamma(\mu k+1)}{2^{1-4\alpha}\,\pi\,\Gamma(2\alpha+j)\Gamma(2\alpha)\Gamma(\alpha+s+\frac{1}{2})\Gamma(k+s+2\alpha+\frac{\nu}{\mu}+1)\Gamma(\mu k+\nu+1)(j-s)!s!}. \tag{34}$$

Combining Eqs. (32)-(34), then we have

$$I^\nu G_i^{(\alpha,\mu)}(t)=\sum_{j=0}^{N}\Delta_\nu(i,j,\alpha,\mu)G_j^{(\alpha,\mu)}(t), \tag{35}$$

where $\Delta_\nu(i,j,\alpha,\mu)$ is given as in Eq. (31).

Finally, we can rewrite Eqs. (35) in a vector form as

$$I^\nu G_i^{(\alpha,\mu)}(t) \simeq \big[\Delta_\nu(i,0,\alpha,\mu),\ \Delta_\nu(i,1,\alpha,\mu),\ \cdots,\ \Delta_\nu(i,N,\alpha,\mu)\big]\Psi_N(t), \quad i=0,1,\cdots,N. \tag{36}$$

Equation (36) completes the proof.

With the aid of properties of FGFs, and after some manipulations, we can state the following corollaries as special cases from Theorem 3.2.

Corollary 3.3 *If $\alpha=0$, we have the shifted fractional Chebyshev functions of the first kind, then $\Delta_\nu(i,j,\mu)$ is given as follows:*

$$\Delta_\nu(i,j,\mu)=\sum_{k=0}^{i}\frac{(-1)^{i-k}\sqrt{\pi}\Gamma(i+k)\Gamma(\mu k+1)}{\Gamma(k+\frac{1}{2})\Gamma(\mu k+\nu+1)(i-k)!k!} \times\sum_{s=0}^{j}\frac{(-1)^{j-s}(2j)j!\Gamma(j+s)\Gamma(k+s+\frac{\nu}{\mu}+\frac{1}{2})}{\Gamma(j)\Gamma(s+\frac{1}{2})\Gamma(k+s+\frac{\nu}{\mu}+1)(j-s)!s!}.$$

Corollary 3.4 *If $\alpha=1$, we have the shifted fractional Chebyshev functions of the second kind, then $\Delta_\nu(i,j,\mu)$ is given as follows:*

$$\Delta_\nu(i,j,\mu)=\sum_{k=0}^{i}\frac{(-1)^{i-k}\sqrt{\pi}(i+1)\Gamma(i+k+2)\Gamma(\mu k+1)}{\Gamma(k+\frac{3}{2})\Gamma(\mu k+\nu+1)(i-k)!k!} \times\sum_{s=0}^{j}\frac{(-1)^{j-s}(j+1)j!\Gamma(j+s+2)\Gamma(k+s+\frac{\nu}{\mu}+\frac{3}{2})}{\Gamma(j+2)\Gamma(s+\frac{3}{2})\Gamma(k+s+\frac{\nu}{\mu}+3)(j-s)!s!}.$$

.

Corollary 3.5 *If $\alpha=\frac{1}{2}$, we have the shifted fractional Legendre functions, then $\Delta_\nu(i,j,\mu)$ is given as follows:*

$$\Delta_\nu(i,j,\mu)=\sum_{k=0}^{i}\frac{(-1)^{i-k}\Gamma(i+k+1)\Gamma(\nu k+1)}{\Gamma(k+1)\Gamma(\mu k+\nu+1)(i-k)!k!} \times\sum_{s=0}^{j}\frac{(-1)^{j-s}(2j+1)\Gamma(j+s+1)\Gamma(k+s+\frac{\nu}{\mu}+1)}{\Gamma(s+1)\Gamma(k+s+\frac{\nu}{\mu}+2)(j-s)!s!}.$$

4. ERROR BOUNDS

In the following Theorem, we introduce an upper bound for estimating the error.

Theorem 4.1 *Suppose that $\mu \in (0,1]$, $D^{i\mu}y(t) \in C(0,1]$ for $i = 0,1,\ldots,N+1$, and*

$$\mathbb{G}_N^{(\alpha,\mu)} = Span\left\{G_0^{(\alpha,\mu)}(t), G_1^{(\alpha,\mu)}(t), \ldots, G_N^{(\alpha,\mu)}(t)\right\}.$$

If $y_N(t) = C^T\Psi_N(t)$ is the best approximation to $y(t)$ from $\mathbb{G}_N^{(\alpha,\mu)}$, then the error bound is given by

$$\|y(t) - y_N(t)\|_2 \le \frac{K_\mu}{\Gamma((N+1)\mu+1)}\sqrt{\frac{\Gamma(\theta+1/2)\Gamma(5/2+2N+\alpha)}{\Gamma(3+2N+\alpha)}},$$

where $\left|D^{\mu(N+1)}y(t)\right| \le K_\mu,\ t \in (0,1]$.

Proof. Since $y_N(t)$ is the best approximation to $y(t)$ from $\mathbb{G}_N^{(\alpha,\mu)}$, then by the definition of the best approximation, we have

$$\|y(t) - y_N(t)\|_2 \le \|y(t) - v(t)\|_2, \quad \forall\, v(t) \in \mathbb{G}_N^{(\alpha,\mu)}.$$

We consider the generalized Taylor formula $y_N(t) = \sum\limits_{i=0}^{N} \frac{t^{i\mu}}{\Gamma(i\mu+1)} D^{i\nu}y(0^+)$ from which we obtain that

$$\left|y(t) - \sum_{i=0}^{N} \frac{t^{i\mu}}{\Gamma(i\mu+1)} D^{i\mu}y(0^+)\right| \le K_\mu \frac{t^{(N+1)\mu}}{\Gamma((N+1)\mu+1)}.$$

Since $y_N(t) \in \mathbb{G}_N^{(\alpha,\mu)}$, we obtain

$$\begin{aligned}
\|y(t) - y_N(t)\|_2^2 &\le \left\|y(t) - \sum_{k=0}^{N} \frac{t^{i\mu}}{\Gamma(i\mu+1)} D^{i\mu}y(0^+)\right\|_2^2 \\
&\le \frac{K_\mu^2}{\Gamma((N+1)\mu+1)^2}\int_0^1 t^{2(N+1)\mu} w^{(\alpha,\mu)}(t)dt \qquad (37)\\
&\le \frac{K_\mu^2\,\Gamma(\theta+1/2)\Gamma(5/2+2N+\alpha)}{\Gamma((N+1)\mu+1)^2\Gamma(3+2N+\alpha)}.
\end{aligned}$$

By taking the square roots, the theorem can be proved.

□

Theorem 4.2 *Suppose that $\mu \in (0,1]$, $\nu > 0$, $\mathrm{G}_{k,i} = \frac{(-1)^{i-k}\Gamma(2\alpha+i+k)\Gamma(\alpha+\frac{1}{2})}{\Gamma(2\alpha)\Gamma(\alpha+k+\frac{1}{2})(i-k)!k!}$ and $e_i = \left\|t^{\mu i+\nu} - \sum\limits_{r=0}^{N} c_{ri}G_r^{(\alpha,\mu)}(t)\right\|_2$, $e_{\boldsymbol{I}^{(\nu)}} = \max\limits_k e_k$, $\left|D^{\mu(N+1)}t^{i\mu+\nu}\right| \le K'_\mu$, $t(0,1]$ for*

$i = 0, 1, \ldots, N$ *and* $\boldsymbol{I}_i^{(\nu)}$ *is the* i*th row of* $\boldsymbol{I}^{(\nu)}$*. Then the error bound of* $\boldsymbol{I}^{(\nu)}$ *is given by*

$$\left\| I^{\nu} G_i^{(\alpha,\mu)}(t) - \boldsymbol{I}_i^{(\nu)} \Psi_N(t) \right\|_2 \leq \frac{K'_\mu}{\Gamma((N+1)\mu+1)} \sqrt{\frac{\Gamma(\theta+1/2)\Gamma(5/2+2N+\alpha)}{\Gamma(3+2N+\alpha)}}. \tag{38}$$

Proof. According to Theorem 4.1, we obtain

$$e_{\mathbf{I}^{(\nu)}} \leq \frac{K'_\mu}{\Gamma((N+1)\mu+1)} \sqrt{\frac{\Gamma(\theta+1/2)\Gamma(5/2+2N+\alpha)}{\Gamma(3+2N+\alpha)}}. \tag{39}$$

Using (32) and (35), we have

$$I^{\nu} G_i^{(\alpha,\mu)}(t) = \sum_{k=0}^{i} \mathrm{G}_{k,i} \frac{\Gamma(\mu k+1)}{\Gamma(\mu k+\nu+1)} t^{\mu k+\nu}, \tag{40}$$

and

$$\mathbf{I}_i^{(\nu)} \Psi_N(t) = \sum_{k=0}^{i} \mathrm{G}_{k,i} \frac{\Gamma(\mu k+1)}{\Gamma(\mu k+\nu+1)} \sum_{j=0}^{N} \eta_{k,j} G_j^{(\alpha,\mu)}(t). \tag{41}$$

So we can write

$$\left\| I^{\nu} G_i^{(\alpha,\mu)}(t) - \mathbf{I}_i^{(\nu)} \Psi_N(t) \right\|_2 \leq \sum_{k=0}^{i} \mathrm{G}_{k,i} \frac{\Gamma(\mu k+1)}{\Gamma(\mu k+\nu+1)} \left\| t^{\mu k+\nu} - \sum_{j=0}^{N} \eta_{k,j} G_j^{(\alpha,\mu)}(t) \right\|_2. \tag{42}$$

Since $\frac{\Gamma(\mu k+1)}{\Gamma(\mu k+\nu+1)} < 1$, we have

$$\left\| I^{\nu} G_i^{(\alpha,\mu)}(t) - \mathbf{I}_i^{(\nu)} \Psi_N(t) \right\|_2 \leq \sum_{k=0}^{i} \mathrm{G}_{k,i} e_k \leq e_{\mathbf{I}^{(\nu)}} \sum_{k=0}^{i} \mathrm{G}_{k,i}. \tag{43}$$

For $\nu \leq 1/2$,

$$\left\| I^{\nu} G_i^{(\alpha,\mu)}(t) - \mathbf{I}_i^{(\nu)} \Psi_N(t) \right\|_2 \leq e_{\mathbf{I}^{(\nu)}}, \tag{44}$$

therefore, we have

$$\left\| I^{\nu} G_i^{(\alpha,\mu)}(t) - \mathbf{I}_i^{(\nu)} \Psi_N(t) \right\|_2 \leq \frac{K'_\mu}{\Gamma((N+1)\mu+1)} \sqrt{\frac{\Gamma(\theta+1/2)\Gamma(5/2+2N+\alpha)}{\Gamma(3+2N+\alpha)}}. \tag{45}$$

5. FRACTIONAL VARIATIONAL PROBLEMS

We consider the FVP

$$\text{Minimize} \quad J = \int_0^1 F\big(t, x(t), D^\nu x(t)\big)dt, \qquad 0 < \nu \le 1, \tag{46}$$

subjected to the initial condition,

$$x(0) = a. \tag{47}$$

First, we approximate $D^\nu x(t)$ by the FGFs $G_k^{(\alpha,\mu)}(t)$ as

$$D^\nu x(t) \simeq \mathbf{X}^T \Psi_N(t), \tag{48}$$

where **X** is an unknown coefficient matrix that can be written as

$$\mathbf{X}^T = [x_0, x_1, \cdots, x_N]. \tag{49}$$

Using (9), we have

$$I^\nu D^\nu x(t) = x(t) - x(0), \tag{50}$$

also adopting Eq. (29) together with Eq. (48), we get

$$I^\nu D^\nu x(t) \simeq \mathbf{X}^T \mathbf{I}^{(\nu)} \Psi_N(t). \tag{51}$$

Using the two previous equations, it is easy to write

$$x(t) \simeq \mathbf{X}^T \mathbf{I}^{(\nu)} \Psi_N(t) + x(0). \tag{52}$$

By approximating $x(0)$ in terms of FGFs $G_k^{(\alpha,\mu)}(t)$ as

$$x(0) \simeq \mathbf{A}^T \Psi_N(t), \tag{53}$$

where

$$\mathbf{A}^T = [a, 0, \cdots, 0]. \tag{54}$$

the function $x(t)$ can be approximated as

$$x(t) \simeq \big(\mathbf{X}^T \mathbf{I}^{(\nu)} + \mathbf{A}^T\big)\Psi_N(t). \tag{55}$$

Using (48) and (55), we can approximate the performance index (46) as

$$J_N[x_0, x_1, \cdots, x_N] \simeq \int_0^1 F\Big(t, \big(\mathbf{X}^T \mathbf{I}^{(\nu)} + \mathbf{A}^T\big)\Psi_N(t), \mathbf{X}^T \Psi_N(t)\Big)dt. \tag{56}$$

The necessary conditions for the optimality of the performance index (46) subjected to the initial condition (47) are

$$\frac{\partial J_N}{\partial x_0} = 0, \qquad \frac{\partial J_N}{\partial x_1} = 0, \qquad \cdots, \qquad \frac{\partial J_N}{\partial x_N} = 0. \tag{57}$$

The system of algebraic equations introduced above can be solved using any standard iteration technique for the unknown coefficients $x_j,\ j = 0, 1, \cdots, N$. Consequently, **X** given in (49) can be calculated.

6. FRACTIONAL OPTIMAL CONTROL PROBLEMS

We consider the FOCP

$$\text{Minimize} \quad J[x,u] = \int_0^1 F\big(t, x(t), u(t)\big)dt, \tag{58}$$

subjected to the dynamic constraint,

$$D^\nu x(t) = g\big(t, x(t)\big) + b(t)u(t), \qquad 0 < \nu \leq 1, \quad b(t) \neq 0, \tag{59}$$

and the initial condition,

$$x(0) = a. \tag{60}$$

First, we can extract the control variable $u(t)$ from the dynamic constraint (59) as

$$u(t) = \frac{1}{b(t)}\Big(D^\nu x(t) - g\big(t, x(t)\big)\Big), \tag{61}$$

then, it is easy to see that the optimal control problem (58)-(60) is equivalent to the following variational problem

$$Minimize \quad J[x] = \int_0^1 F\Big(t, x(t), \frac{1}{b(t)}\Big(D^\nu x(t) - g\big(t, x(t)\big)\Big)\Big)dt, \tag{62}$$

subject to the initial condition (60).

Following the same technique introduced in the previous Section, we approximate $D^\nu x(t)$ by the FGFs as in Eq. (48) and $x(t)$ as in Eq. (55), and we can rewrite the performance index (62) as in the form

$$\begin{aligned} J_N[x_0, x_1, \cdots, x_N] \simeq \int_0^1 F\Big(t, \big(\mathbf{X}^T\mathbf{I}^{(\nu)} + \mathbf{A}^T\big)\Psi_N(t), \frac{1}{b(t)}\Big(\mathbf{X}^T\Psi_N(t) - \\ g\big(t, (\mathbf{X}^T\mathbf{I}^{(\nu)} + \mathbf{A}^T)\Psi_N(t)\big)\Big)\Big)dt. \end{aligned} \tag{63}$$

The necessary conditions for the optimality of the performance index (58) subjected to the dynamic constraint (59) and the initial condition (60) become,

$$\frac{\partial J_N}{\partial x_0} = 0, \qquad \frac{\partial J_N}{\partial x_1} = 0, \qquad \cdots, \qquad \frac{\partial J_N}{\partial x_N} = 0. \tag{64}$$

The previous system of algebraic equations can be solved easily to obtain the unknown coefficients x_j, $j = 0, 1, \cdots, N$.

Table 1.

Approximate values of J at $N = 8$ with various choices of ν, μ, and α for Example 1.

ν	μ	$\alpha = \frac{1}{4}$	$\alpha = \frac{1}{3}$	$\alpha = \frac{1}{2}$	$\alpha = 1$
1.00	1.00	−0.166667	−0.166667	−0.166667	−0.166597
0.99	0.99	−0.169299	−0.169299	−0.169299	−0.169299
0.90	0.90	−0.193051	−0.193051	−0.193051	−0.193052
0.80	0.80	−0.221686	−0.221686	−0.221687	−0.221690

7. ILLUSTRATIVE TEST PROBLEMS

Example 1 *Consider the following FVP*

$$\textit{Minimize} \quad J = \int_0^1 \Big[\frac{1}{2}\Big(D^\nu x(t)\Big)^2 - x(t)\Big]dt, \tag{65}$$

with the following initial condition

$$x(0) = 0. \tag{66}$$

The exact solution of this problem for $\nu = 1$ is

$$x(t) = t\big(1 - \frac{t}{2}\big). \tag{67}$$

The above FVP has been solved by the Rayleigh-Ritz method as in Sec. 4. Table 1 lists the approximate values of the function J (65) at $N = 8$ with various choices of ν, μ and α. Figure 1 shows the absolute error function (AEF) of the function $x(t)$ at $N = 8$, $\nu = \mu = 1$, and $\alpha = \frac{1}{3}$. In addition, Fig. 2 plots the approximate solution of $x(t)$ as a function of time at $N = 8$ and $\alpha = 1$ with different values of ν and μ, while in Fig. 3 the exact and approximate solutions of $x(t)$ at $\nu = \mu = 1$, $\alpha = \frac{1}{2}$ with $N = 3,\ 6,\ 9$ are shown.

Example 2 *Consider the following FVP [33]*

$$\textit{Minimize} \quad J = \int_0^1 \Big[\Big(D^\nu x(t)\Big)^2 + t\Big(D^\nu x(t)\Big)\Big]dt, \tag{68}$$

with the following initial condition

$$x(0) = 0. \tag{69}$$

The exact solution of this problem for $\nu = 1$ is

$$x(t) = -\frac{t^2}{4}. \tag{70}$$

This problem was considered in [33]. The author used the Haar wavelet method (HWM) for solving this problem at $\nu = 1$, obtained the approximate results of $x(t)$

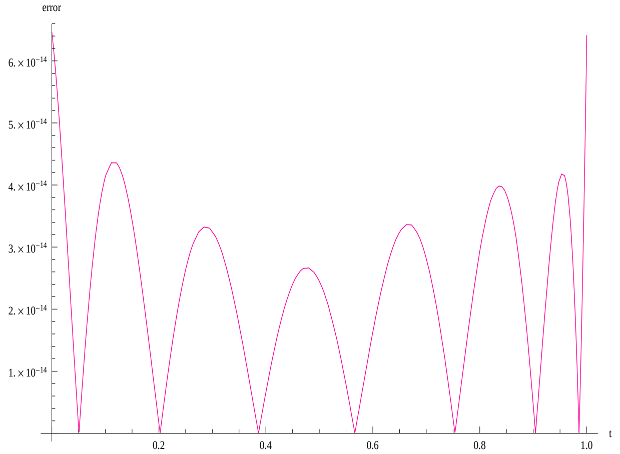


Fig. 1 – AEF of $x(t)$ at $N = 8$, $\nu = \mu = 1$, and $\alpha = \frac{1}{3}$ for Example 1.

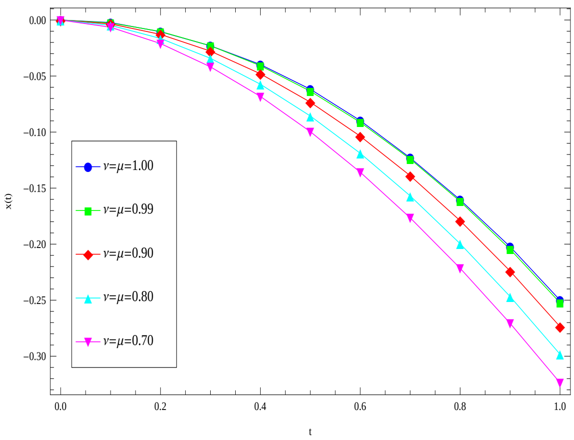


Fig. 2 – Approximate solution of $x(t)$ at $N = 8$ and $\alpha = 1$ with different values of ν and μ for Example 1.

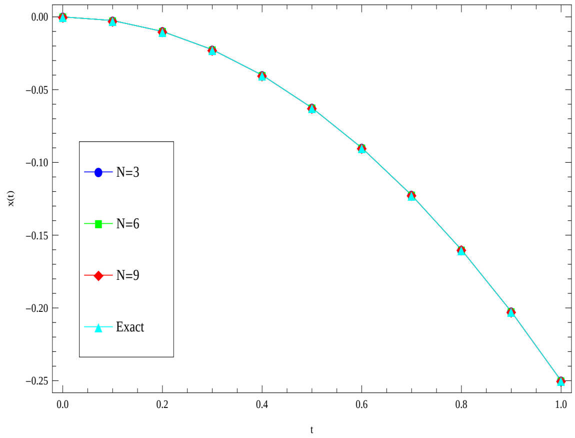


Fig. 3 – Exact and approximate solutions of $x(t)$ at $\nu = \mu = 1$, $\alpha = \frac{1}{2}$ with $N = 3,\ 6,\ 9$ for Example 1.

at some values of t with $N = 8$ and compared the results with its analytic solution, see Table 3 in [33]. Table 2 gives the approximate results of $x(t)$ at $\nu = \mu = 1$ and $N = 8$ with different values of t and α and compare the results with those achieved by HWM in [33]. Figure 5 shows the approximate solution of $x(t)$ at $N = 8$ and $\alpha = 1$ with different values of ν and μ. Finally, in Fig. 4, we plotted the exact and approximate solutions of $x(t)$ at $\nu = \mu = 1$, $\alpha = \frac{1}{2}$, and $N = 3,6,9$.

Example 3 *Consider the following FOCP [18,19,34]*

$$\textit{Minimize} \quad J = \frac{1}{2}\int_0^1 \left[x^2(t) + u^2(t)\right] dt, \tag{71}$$

subject to the dynamical fractional control system

$$D^{\nu} x(t) = -x(t) + u(t), \tag{72}$$

and the initial condition

$$x(0) = 1. \tag{73}$$

The exact solution of this problem, when $\nu = 1$, is given as

$$\begin{aligned} x(t) &= \coth(\sqrt{2}t) - 0.98\ \sinh(\sqrt{2}t), \\ u(t) &= (1 - 0.98\sqrt{2})\cosh(\sqrt{2}t) + (\sqrt{2} - 0.98)\sinh(\sqrt{2}t). \end{aligned} \tag{74}$$

Akbarian and Keyanpour [18] considered this problem and applied the perturbation homotopy method together with the parameterization method (PH-PM) for getting its numerical solution, while the authors in [19] used the Legendre-Gauss

Table 2.

Comparing the approximate results of $x(t)$ between our method and the HWM in [33] at $\nu = \mu = 1$ and $N = 8$ with different values of t and α for Example 2.

t	Our method				HWM in [33]	Analytic solution
	$\alpha = \frac{1}{4}$	$\alpha = \frac{1}{3}$	$\alpha = \frac{1}{2}$	$\alpha = 1$		
0	0	0	0	0	0	0
0.125	-0.0039	-0.0039	-0.0039	-0.0038	-0.0039	-0.0039
0.250	-0.0156	-0.0156	-0.0156	-0.0162	-0.0156	-0.0156
0.375	-0.0352	-0.0352	-0.0352	-0.0351	-0.0352	-0.0352
0.500	-0.0625	-0.0625	-0.0625	-0.0618	-0.0625	-0.0625
0.625	-0.0977	-0.0977	-0.0977	-0.0976	-0.0977	-0.0977
0.750	-0.1406	-0.1406	-0.1406	-0.1413	-0.1406	-0.1406
0.875	-0.1914	-0.1914	-0.1914	-0.1913	-0.1914	-0.1914
1	-0.2500	-0.2500	-0.2500	-0.2500	-0.2539	-0.2500

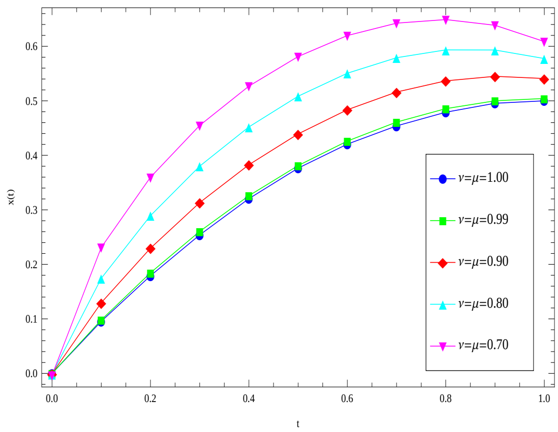


Fig. 4 – Approximate solution of $x(t)$ at $N = 8$ and $\alpha = 1$ with different values of ν and μ for Example 2.

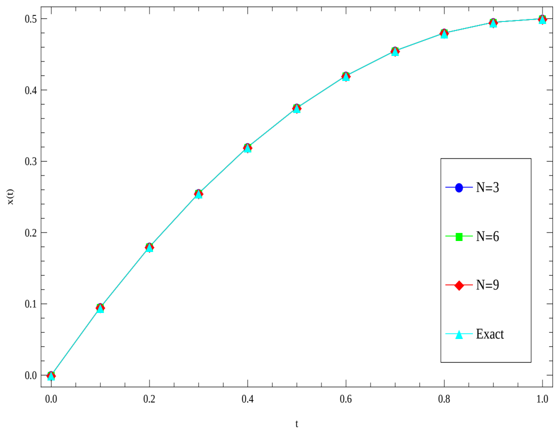


Fig. 5 – Exact and approximate solutions of $x(t)$ at $\nu = \mu = 1$, $\alpha = \frac{1}{2}$, and $N = 3, 6, 9$ for Example 2.

quadrature formula (LGQF) to solve it. Recently, in [34] the authors applied the variational iteration method (VIM) for solving this problem.

In Table 3 we compare the maximum absolute errors (MAEs) of $x(t)$ and $u(t)$ between our numerical approach with the VIM in [34] and the LGQF in [19] at $\alpha = \frac{1}{2}$, $\nu = \mu = 1$ and various choices of N. In Table 4 we compare the results of the approximate value of the cost function J obtained using our numerical approach at different choices of ν, μ and α with those obtained by PH-PM [18] and VIM [34]. Figure 6 presents the approximate solutions of $x(t)$ and $u(t)$ at $N = 8$ and $\alpha = 1$ with different values of ν and μ. Finally, in Fig. 7, we compare the exact and approximate solutions of $x(t)$ and $u(t)$ at $\nu = \mu = 1$, $\alpha = \frac{1}{3}$ with $N = 3, 6, 9$.

Example 4 *Consider the following FOCP [18, 19, 34]*

$$\textit{Minimize} \quad J = \frac{1}{2}\int_0^1 [x^2(t) + u^2(t)]dt, \tag{75}$$

subject to the dynamical fractional control system

$$D^\nu x(t) = tx(t) + u(t), \tag{76}$$

and the initial condition

$$x(0) = 1. \tag{77}$$

In Table 5, comparisons of the approximate value of the cost function J are made between our numerical approach, the PH-PM [18], the VIM [34] and the method in [19] at different choices of ν, μ, and α. Figure 8 shows the approximate solutions of $x(t)$ and $u(t)$ at $N = 8$ and $\alpha = 1$ with different values of ν and μ.

Table 3.

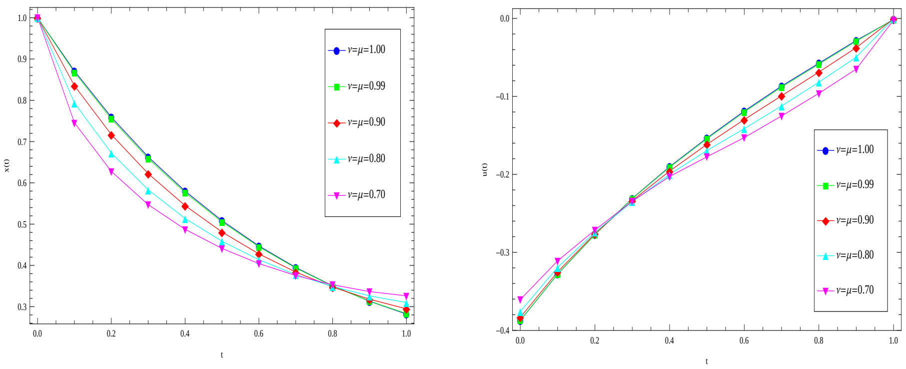
MAEs of $x(t)$ and $u(t)$ achieved using our method, the VIM in [34] and the LGQF in [19] at $\alpha = \frac{1}{2}$, $\nu = \mu = 1$ and various choices of N for Example 3.

	VIM [34]		LGQF [19]		Our method	
	N	$MAEs$	N	$MAEs$	N	$MAEs$
$x(t)$	10	2.18284×10^{-2}	3	1.00212×10^{-3}	2	1.15817×10^{-2}
	30	5.05794×10^{-4}	5	5.05995×10^{-6}	4	8.99562×10^{-5}
	50	4.19787×10^{-6}	7	1.29671×10^{-8}	6	3.11879×10^{-7}
	80	1.63814×10^{-9}	9	2.01182×10^{-11}	8	6.08778×10^{-10}
			11	1.05293×10^{-12}	10	7.46514×10^{-13}
$u(t)$	10	6.76558×10^{-2}	3	5.99596×10^{-3}	2	6.22299×10^{-3}
	30	2.95124×10^{-4}	5	3.31436×10^{-5}	4	4.83361×10^{-5}
	50	1.50618×10^{-6}	7	1.06056×10^{-7}	6	1.67582×10^{-7}
	80	6.18633×10^{-9}	9	1.94284×10^{-10}	8	3.27107×10^{-10}
			11	7.13040×10^{-11}	10	1.43430×10^{-12}

Table 4.

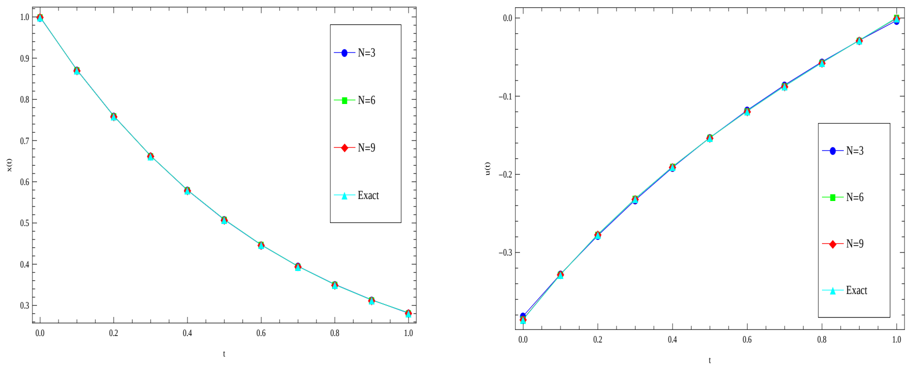
Approximate value of J at different choices of ν, μ, and α for Example 3.

ν	PH-PM [18]	VIM [34]	Our method			
			μ	$\alpha = \frac{1}{4}$	$\alpha = \frac{1}{2}$	$\alpha = 1$
1.00	0.192909	0.192909	1.00	0.192909	0.192909	0.192909
0.99	0.19153	0.19153	0.99	0.19153	0.19153	0.19153
0.90	0.17952	0.17953	0.90	0.17953	0.17953	0.17953
0.80	0.16729	0.16711	0.80	0.16708	0.16708	0.16708



(a) (b)

Fig. 6 – Approximate solutions of $x(t)$ (a) and $u(t)$ (b) at $N = 8$ and $\alpha = 1$ with different values of ν and μ for Example 3.



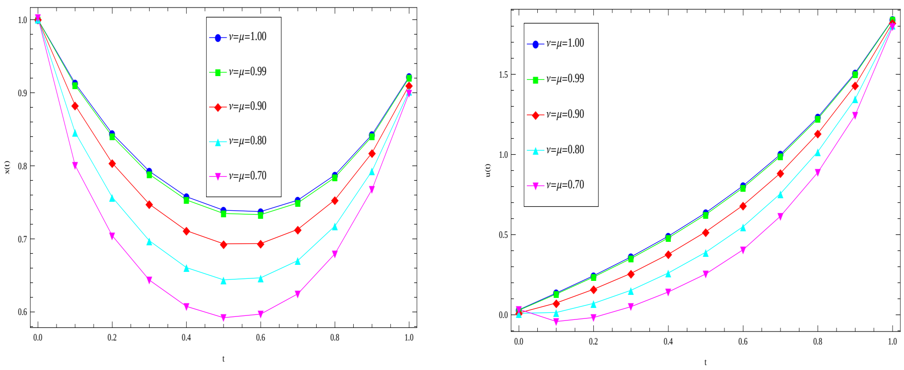
(a) (b)

Fig. 7 – Exact and approximate solutions of $x(t)$ (a) and $u(t)$ (b) at $\nu = \mu = 1$, $\alpha = \frac{1}{3}$ with $N = 3, 6, 9$ for Example 3.

Table 5.

Approximate value of J at different choices of ν, μ, and α for Example 4.

ν	PH-PM [18]	VIM [34]	Method in [19]	Our method			
				μ	$\alpha = \frac{1}{4}$	$\alpha = \frac{1}{2}$	$\alpha = 1$
1.00	0.48427	0.48426	0.48426	1.00	0.48427	0.48427	0.48427
0.99	0.48347	0.48346	0.48346	0.99	0.48346	0.48346	0.48346
0.90	0.47605	0.47593	0.47588	0.90	0.47588	0.47588	0.47588
0.80	0.46722	0.46722	0.46697	0.80	0.46698	0.46698	0.46698



(a) (b)

Fig. 8 – Approximate solutions of $x(t)$ (a) and $u(t)$ (b) at $N = 8$ and $\alpha = 1$ with different values of ν and μ for Example 4.

8. CONCLUSION

In this paper, we developed an efficient numerical approach for solving a class of FVPs and FOCPs. The operation matrix of fractional integrals of the FGFs was derived and applied together with the Rayleigh-Ritz method to obtain systems of algebraic equations in the unknown expansion coefficients that may be evaluated easily using any iterative method. Several numerical examples were given and the numerical solutions achieved by the present techniques were compared with the exact solutions and recent numerical techniques reported in the literature.

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