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SOLUTION OF THE HYPERBOLIC KEPLER EQUATION BY ADOMIAN'S ASYMPTOTIC DECOMPOSITION METHOD

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Abstract. The hyperbolic Kepler equation is of practical interest in astronomy. It is often used to describe the eccentric anomaly of a comet of extrasolar origin in its hyperbolic trajectory past the Sun. Efficient determination of the radial distance and/or the Cartesian coordinates of the comet requires accurate calculation of the eccentric anomaly, hence the need for a convenient, robust method to solve Kepler's equation of hyperbolic type. In this paper, the Adomian's asymptotic decomposition method is proposed to solve this equation. Our calculations have demonstrated a rapid rate of convergence of the sequence of the obtained approximate solutions, which are displayed in several graphs. Also, we have shown in this paper that only a few terms of the Adomian decomposition series are sufficient to achieve extremely accurate numerical results even for much higher values than those in the literature for the mean anomaly and the eccentricity of the orbit. The main characteristic of the obtained approximate solutions is that they are all odd functions in the mean anomaly, which we have illustrated through graphs. In addition, it is found that the absolute remainder error using only three components of Adomian's solution decreases across a specified domain and approaches zero as the eccentric anomaly tends to infinity. Moreover, the absolute remainder error decreases by increasing the number of components of the Adomian decomposition series. Finally, the current analysis may be the first to make an effective application of the Adomian's asymptotic decomposition method in astronomical physics.

Key words: Adomian's asymptotic decomposition method; Adomian polynomials; hyperbolic Kepler's equation; series solution.

1. INTRODUCTION

In celestial mechanics, Kepler's equations of elliptical and hyperbolic types play important roles. Efficient determination of the accurate position of an object (a planet, comet or an asteroid) when orbiting the Sun requires solving either of these equations with a solution method of high accuracy. In our solar system, some comets of extrasolar origin follow hyperbolic trajectories past the Sun. Such comets enter the solar system coming from the Oort cloud and interstellar space and may exit the solar system through hyperbolic trajectories. In astronomy, the original orbits of such comets may change from elliptical to hyperbolic, especially, when taking into account the possible gravitational attraction of the planets of substantial mass, *e.g.*, Jupiter. In this paper, Kepler's equation of hyperbolic type is considered in the standard form [1],

$$e\sinh\big(H(t)\big) - H(t) = M(t), \quad 1 \le e < \infty, \quad 0 \le M < \infty, \tag{1}$$

where H is defined as the eccentric anomaly, $M(t) = \sqrt{\frac{\mu}{a^3}}(t-\tau)$ is the mean anomaly, a is the semi-major axis, e is the eccentricity of the orbit, $\mu = GM$ is the gravitational parameter of the central body of mass M, where G is the universal gravitational constant and τ is the time of passage through the closest point of approach to the focus. The polar equation of a hyperbola with its focus at the origin may be written as

$$r = \frac{a(e^2-1)}{1+e\cos f}. \tag{2}$$

This angle f is given in terms of H through the following relationship [1],

$$\tan\left(\frac{f}{2}\right) = \sqrt{\frac{e+1}{e-1}}\tan\left(\frac{H}{2}\right). \tag{3}$$

In addition, at an instant t, the (x, y) coordinates of an object in the standard frame of reference (where the central body is at the origin and the x-axis points towards the periapsis) are given in terms of H by

$$x = a\big(e - \cosh H\big), \tag{4}$$

$$y = a\sqrt{e^2-1}\sinh H. \tag{5}$$

So, in order to calculate the radial distance r and the true anomaly f of a comet when orbiting the Sun at a specified time t, the hyperbolic Kepler equation (1) is first solved for H at that time t and then Eqs. (2) and (3) are applied. As shown by Eq. (1), the hyperbolic Kepler equation is a transcendental equation that has no exact closed-form analytic solution. Although many authors [1–10] have devised various numerical and analytical solutions for Kepler's equation of elliptical type, little effort has been devoted to investigate the hyperbolic form of this equation [11–14].

Searching for a new accurate but simple analytical solution for the hyperbolic Kepler equation is still of manifest practical interest. In order to contribute to an improved solution of this problem, the authors believe that the

The Adomian's decomposition method (ADM) can be effectively applied to solve the hyperbolic form of this equation. The ADM is a systematic analytic approximation method for solving algebraic and transcendental equations, matrix equations, nonlinear integral equations, and nonlinear differential equations including both nonlinear initial value problems and nonlinear boundary value problems even for irregular boundary contours. It has been widely implemented to solve a large number of frontier problems in the applied sciences and engineering and can be also extended to cover many scientific models [15–42]. It expresses the solution in the form of an infinite series. Under physically appropriate conditions, this series often rapidly converges and hence a few terms of Adomian's method are sufficient to obtain accurate numerical results for the investigated problem. In this case, the sequence of approximate solutions by Adomian's method converges to a certain curve or function. For example, the ADM has been applied by Ebaid [31] to solve the Thomas-Fermi equation that has no exact solution. In that paper, he showed geometrically that the sequence of the approximate solutions converges to a certain curve, which may even be the exact solution for that problem.

The objective of this paper is to analyze the hyperbolic Kepler equation by using the ADM. We show that the Adomian's decomposition series solution to the current problem closely coincides with those in the literature for all values of the mean anomaly parameter $M \in [0,\infty)$ and for all values of the eccentricity e of the orbit using only a few of Adomian's solution components. In addition, it will be shown in this paper that the sequence of the Adomian's decomposition approximate solutions converges rapidly in a much wider range than those considered in the literature for the parameters M and e.

2 APPLICATION OF ADOMIAN'S ASYMPTOTIC DECOMPOSITION METHOD

The Adomian's asymptotic decomposition method (AADM) is applied in this Section to calculate a sequence of approximate analytic solutions for the hyperbolic Kepler equation. We rewrite Eq. (1) in the canonical form as

$$\sinh\big(H(t)\big) = \frac{M(t)}{e} + \frac{1}{e}H(t). \tag{6}$$

The series of the Adomian polynomials and the Adomian decomposition series are

$$\begin{aligned} &\sinh\big(H(t)\big) = \sum_{n=0}^{\infty} A_n(t),\ \ A_n(t) = A_n\big(H_0(t),\ldots,H_n(t)\big), \\ &H(t) = \sum_{n=0}^{\infty} H_n(t). \end{aligned} \tag{7}$$

Upon substitution of Eq. (7) into Eq. (6), we obtain

$$\sum_{n=0}^{\infty} A_n(t) = \frac{M(t)}{e} + \frac{1}{e}\sum_{n=0}^{\infty} H_n(t). \tag{8}$$

Accordingly, the following algorithm can be established using the Adomian recursion scheme,

$$A_0(t) = \frac{M(t)}{e}, \quad A_{n+1}(t) = \frac{1}{e} H_n, \quad n \geq 0. \tag{9}$$

The Adomian polynomials for the hyperbolic sine nonlinearity were defined by Adomian and Rach in 1983 [22] as

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\sinh\left(\sum_{m=0}^{\infty} \lambda^m H_m(t)\right)\right]_{\lambda=0}. \tag{10}$$

Applying this formula, the first several Adomian polynomials for the hyperbolic sine nonlinearity are

$$\begin{aligned}
A_0 &= \sinh(H_0),\\
A_1 &= H_1 \cosh(H_0),\\
A_2 &= H_2 \cosh(H_0) + \frac{1}{2!} H_1^2 \sinh(H_0),\\
A_3 &= H_3 \cosh(H_0) + H_1 H_2 \sinh(H_0) + \frac{1}{3!} H_1^3 \cosh(H_0),\\
&\;\;\vdots
\end{aligned} \tag{11}$$

Hence

$$\begin{aligned}
A_1 &= \frac{1}{e} H_0,\\
A_2 &= \frac{1}{e} H_1,\\
A_3 &= \frac{1}{e} H_2,\\
A_4 &= \frac{1}{e} H_3,\\
&\;\;\vdots
\end{aligned} \tag{12}$$

and so on. Combining (11) and (12), we obtain

$$\begin{aligned}
&\sinh(H_0) = \frac{M}{e},\\
&H_1 \cosh(H_0) = \frac{1}{e} H_0,\\
&H_2 \cosh(H_0) + \frac{1}{2!} H_1^2 \sinh(H_0) = \frac{1}{e} H_1,\\
&H_3 \cosh(H_0) + H_1 H_2 \sinh(H_0) + \frac{1}{3!} H_1^3 \cosh(H_0) = \frac{1}{e} H_2.
\end{aligned} \tag{13}$$

Therefore

$$\begin{aligned}
H_0 &= \sinh^{-1}\left(\frac{M}{e}\right),\\
H_1 &= \frac{\sinh^{-1}\left(\frac{M}{e}\right)}{\sqrt{e^2+M^2}},\\
H_2 &= \frac{2(e^2+M^2)\sinh^{-1}\left(\frac{M}{e}\right) - M\sqrt{e^2+M^2}\left(\sinh^{-1}\left(\frac{M}{e}\right)\right)^2}{2(e^2+M^2)^2},\\
H_3 &= \frac{6(e^2+M^2)\sinh^{-1}\left(\frac{M}{e}\right) - 9M\sqrt{e^2+M^2}\left(\sinh^{-1}\left(\frac{M}{e}\right)\right)^2 - (e^2-2M^2)\left(\sinh^{-1}\left(\frac{M}{e}\right)\right)^3}{6(e^2+M^2)^{5/2}},\\
&\ \ \vdots
\end{aligned} \tag{14}$$

The ADM gives the n-term approximate analytic solution $\Phi_n(t)$ for the hyperbolic Kepler equation as

$$\Phi_n(t) = \sum_{i=0}^{n-1} H_i(t). \tag{15}$$

Accordingly, we calculate

$$\Phi_2(t) = \left(1 + \frac{1}{\sqrt{e^2+M^2}}\right)\sinh^{-1}\left(\frac{M}{e}\right),$$

$$\Phi_3(t)=\left(1+\frac{1}{\sqrt{e^2+M^2}}+\frac{1}{e^2+M^2}\right)\sinh^{-1}\left(\frac{M}{e}\right)-\frac{M}{2(e^2+M^2)^{3/2}}\sinh^{-1}\left(\frac{M}{e}\right)^2, \tag{16}$$

$$\Phi_4(t)=\left(1+\frac{1}{\sqrt{e^2+M^2}}+\frac{1}{e^2+M^2}+\frac{1}{(e^2+M^2)^{3/2}}\right)\sinh^{-1}\left(\frac{M}{e}\right)-\frac{2M}{(e^2+M^2)^{3/2}}$$
$$\left(\sinh^{-1}\left(\frac{M}{e}\right)\right)^2-\frac{(e^2-2M^2)}{6(e^2+M^2)^{5/2}}\left(\sinh^{-1}\left(\frac{M}{e}\right)\right)^3.$$

In a subsequent Section, we will show that the sequence of the approximate solutions in (16) for the hyperbolic Kepler equation is convergent in a wider range than those reported in the literature for M and e. In addition, the accuracy of the present numerical results will be validated by calculating the absolute remainder error $|RE_{n+1}(t)|$ defined by

$$\left|RE_{n+1}(t)\right|=\left|e\sinh\left(\Phi_{n+1}(t)\right)-\Phi_{n+1}(t)-M\right|,\quad n\geq 0, \tag{17}$$

by using the n-term approximate solution to estimate the eccentric anomaly H. Moreover, the advantage and the effectiveness of the present low-order approximate analytic solutions for the hyperbolic Kepler equation over several existing methods in the literature will be proved for certain higher values of the parameters M and e.

3. DISCUSSION

In the previous Section, Adomian's asymptotic decomposition method has been applied to obtain the approximate solutions of the hyperbolic Kepler equation. Such approximate solutions are applied in this discussion to obtain several plots. Let us begin by graphically demonstrating the convergence of the present approximate solutions. In Fig. 1, the approximate solutions $\Phi_3(t)$, $\Phi_5(t)$, and $\Phi_7(t)$ are plotted for $e=1.5$ versus the mean anomaly M. A rapid convergence is observed in this figure using only a few terms of the Adomian asymptotic solutions.

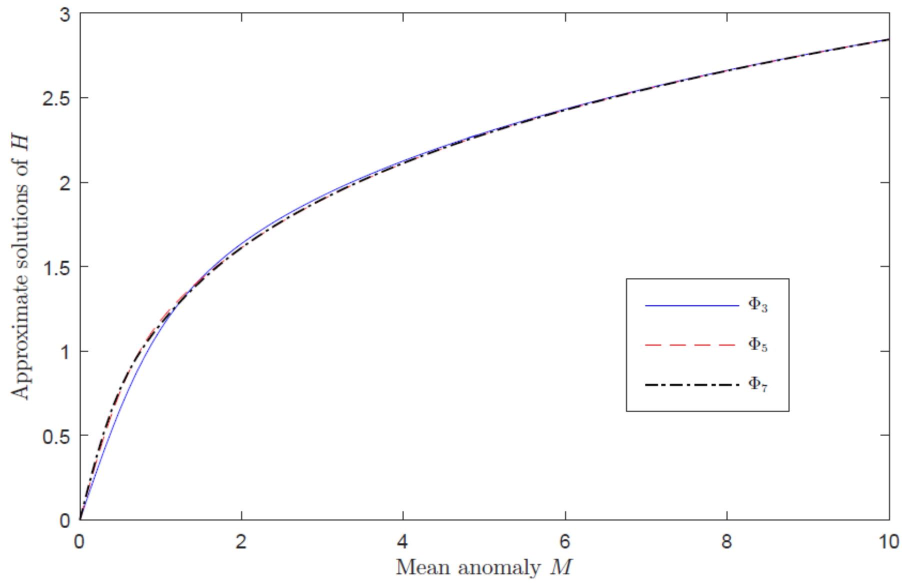

Figure 1: Convergence of the approximate solutions versus *M* at *e* = 1.5

The main result here is that the rate of convergence is increased for higher values of M, where at $M \geq 4$ the three-term approximate solution $\Phi_3(t)$ of Adomian's asymptotic decomposition method is sufficient to provide a remarkably accurate solution, while at the lower values of M in the domain $[0,4)$ a higher-order approximate solution such as $\Phi_n(t)$ for $n \geq 5$ is required to achieve a similarly high accuracy. In addition, these approximate solutions are all odd functions in the mean anomaly M which has been shown in Fig. 2 for a selected higher value of the eccentricity parameter $e = 10.5$.

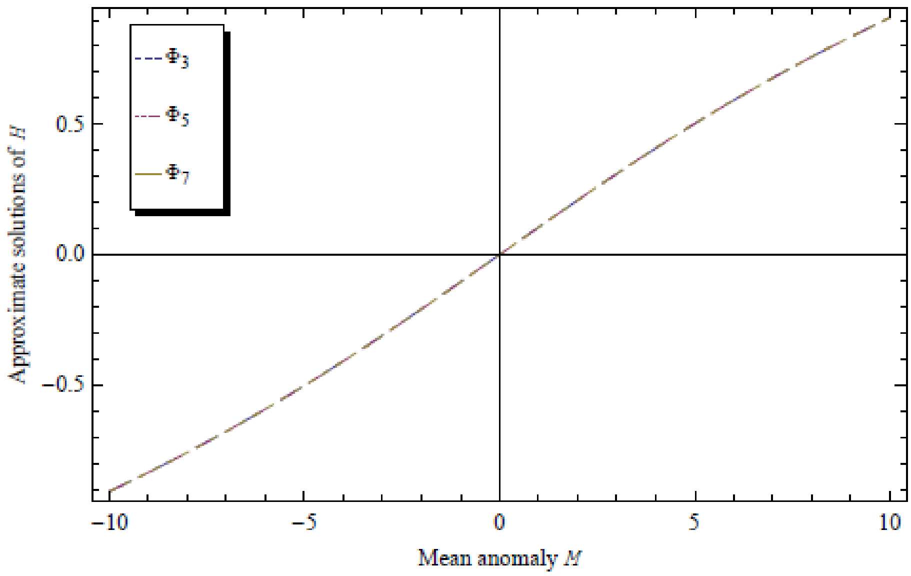

Figure 2: The odd property of the approximate solutions versus *M* at *e* = 10.5.

In a wider range of $M \in [0,100]$, it has been also shown from Fig. 3 that the approximate solutions $\Phi_3(t)$, $\Phi_5(t)$, and $\Phi_7(t)$ converge to a certain curve/function. Besides, the odd property is also shown in Fig. 4 for a further wider range $M \in [-100,100]$.

Moreover, Fig. 5 and Fig. 6 also demonstrate the rapid rate of convergence of Adomian's sequence of analytic approximate solutions versus the eccentricity parameter in the domain $e \in [1.5,25.5]$ at the lowest and the highest values of M, which was considered by Sharaf *et al.* [13], respectively. It is readily apparent from these figures that Adomian's solutions also converge, even at the highest considered value, $M = 175000.5$. The aforementioned discussion corroborates the effectiveness of Adomian's asymptotic decomposition method in quickly and accurately solving the hyperbolic Kepler equation.

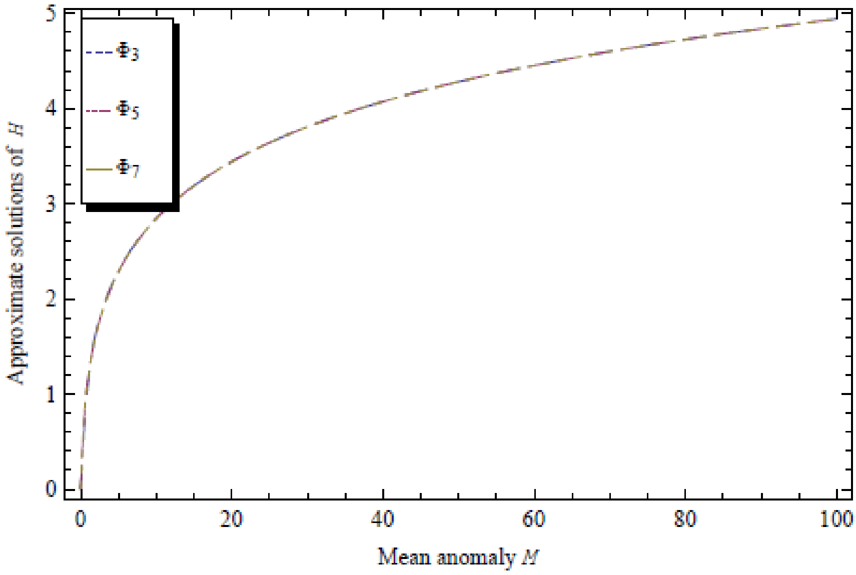

Figure 3: Convergence of the approximate solutions versus M at e = 1.5 in a wider range.

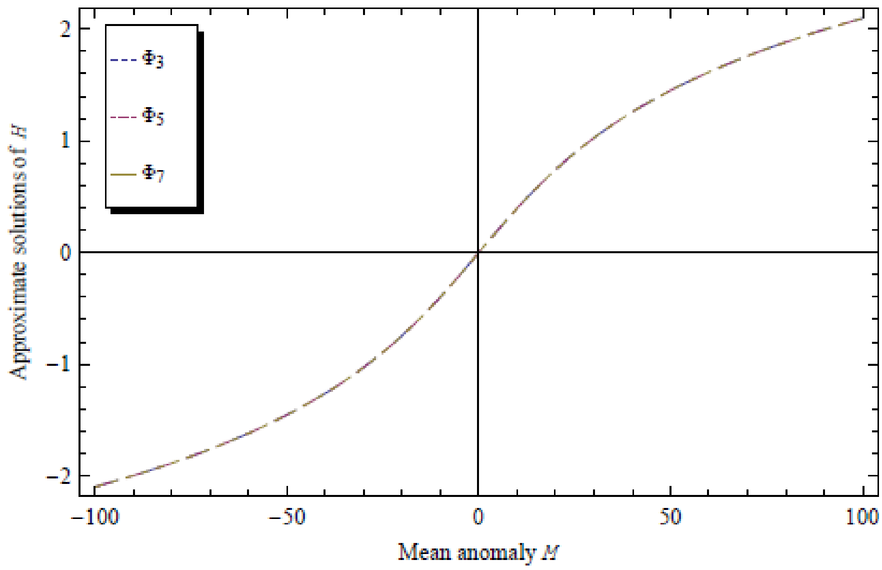

Figure 4: The odd property of the approximate solutions versus M at e = 10.5 in a wider range.

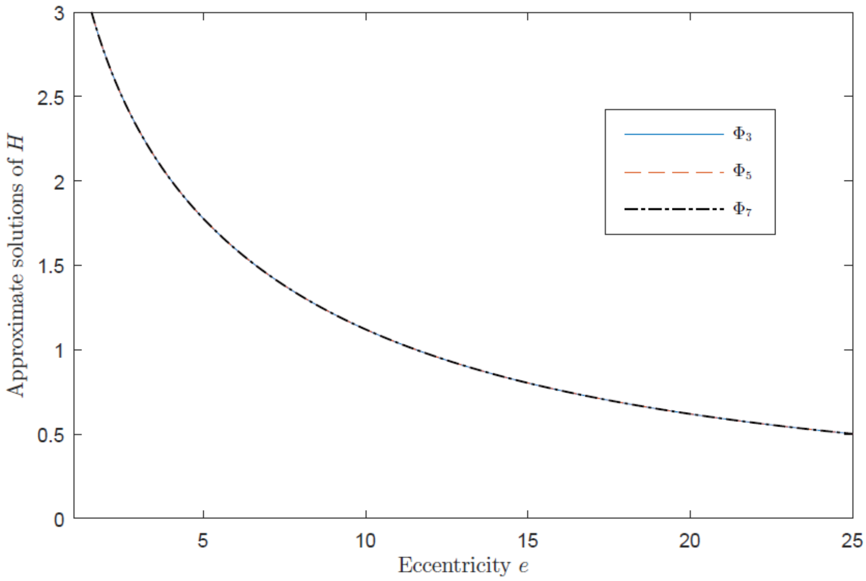

Figure 5: Convergence of the approximate solutions versus *e* at *M* = 12.58.

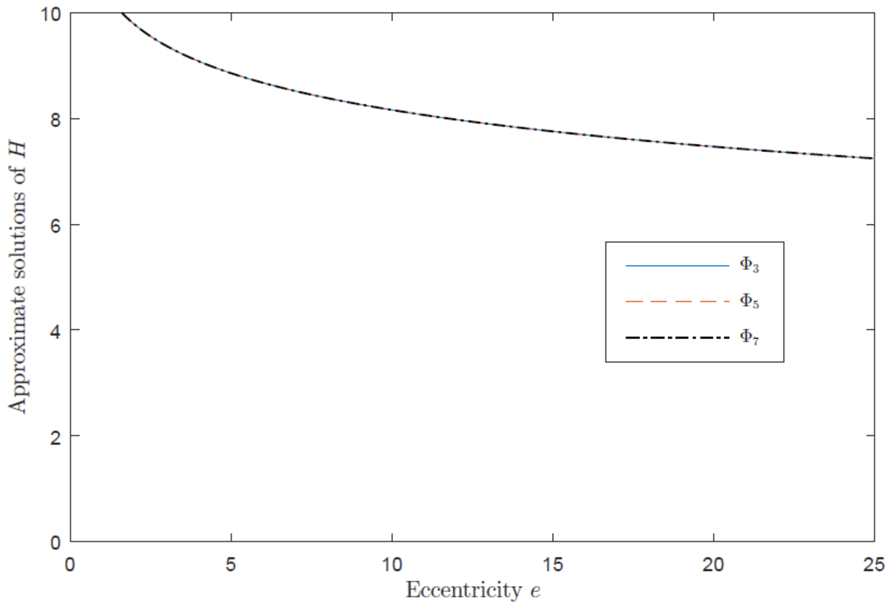

Figure 6: Convergence of the approximate solutions versus *e* at *M* = 17500.5.

The effectiveness and efficiency of the current lower-order approximate solutions are numerically validated in Figs. 7-8. In these figures, the corresponding absolute remainder errors $|RE_3|$, $|RE_5|$, and $|RE_7|$ are displayed at two selected values of the eccentricity $e = 1.5$ and $e = 100$ for two selected domains of the mean anomaly $M \in [0,100]$ in Fig. 7 and $M \in [0,10000]$ in Fig. 8. We conclude from figures 7-8 that

• the absolute remainder error $|RE_3|$ using only three components of Adomian's asymptotic decomposition method decreases and approaches zero as M tends to infinity. However, the maximum value of the absolute remainder error $|RE_3|$ is about 0.1 radians in the region $M \in (0,3)$ and it is about 0.08 radians $M \in (3,6)$ as in Fig. 7 and then it decreases with increasing M.

• the absolute remainder error decreases with increasing the eccentricity e. For example, the maximum value of the absolute remainder error $|RE_3|$ is about 0.000025 radians as in Fig. 8 when $e = 100$.

• the absolute remainder error decreases by increasing the number of components as illustrated by the curves of $|RE_5|$ and $|RE_7|$ in Figs. 7-8.

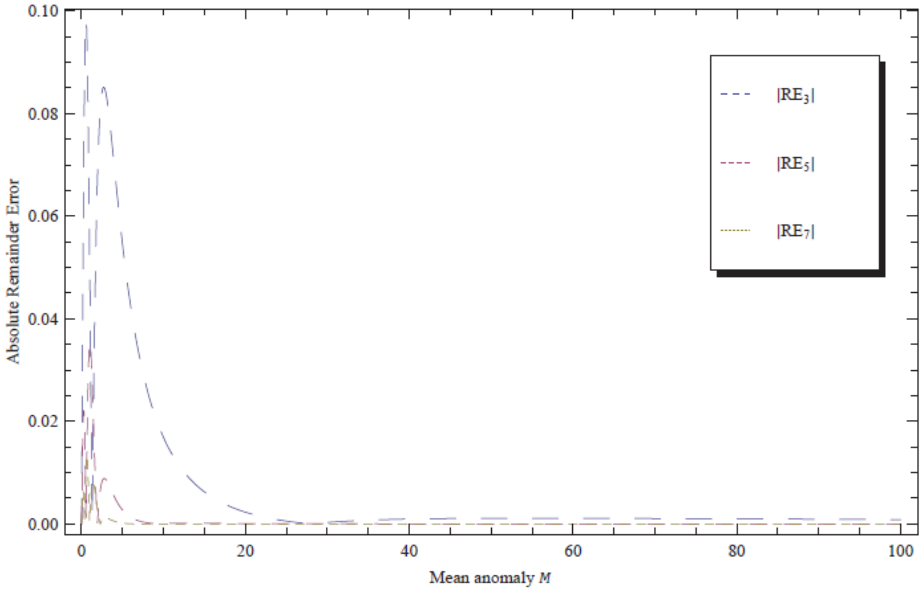

Figure 7: The absolute remainder error versus M at e = 1.5.

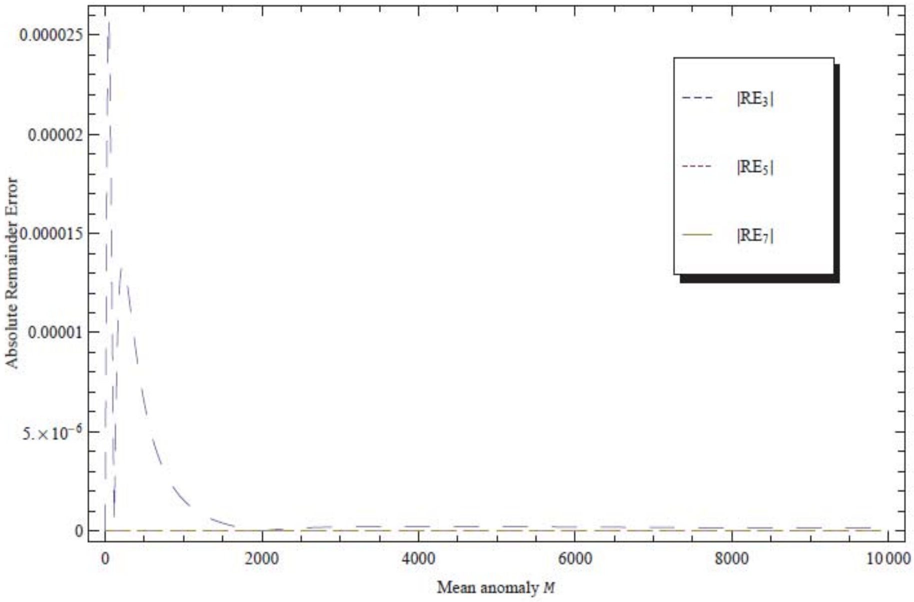

Figure 8: The absolute remainder error versus M at e = 100.

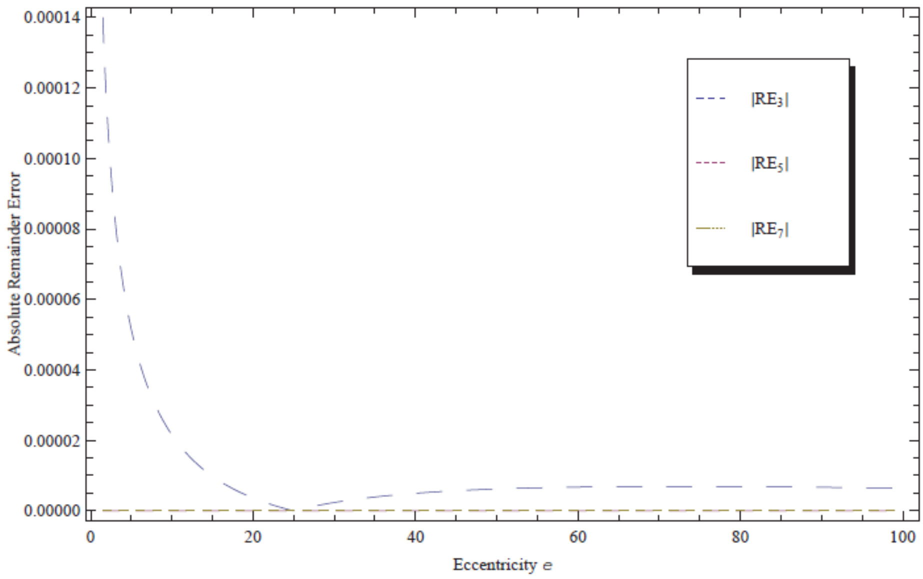

Figure 9: The absolute remainder error versus e at M = 500.

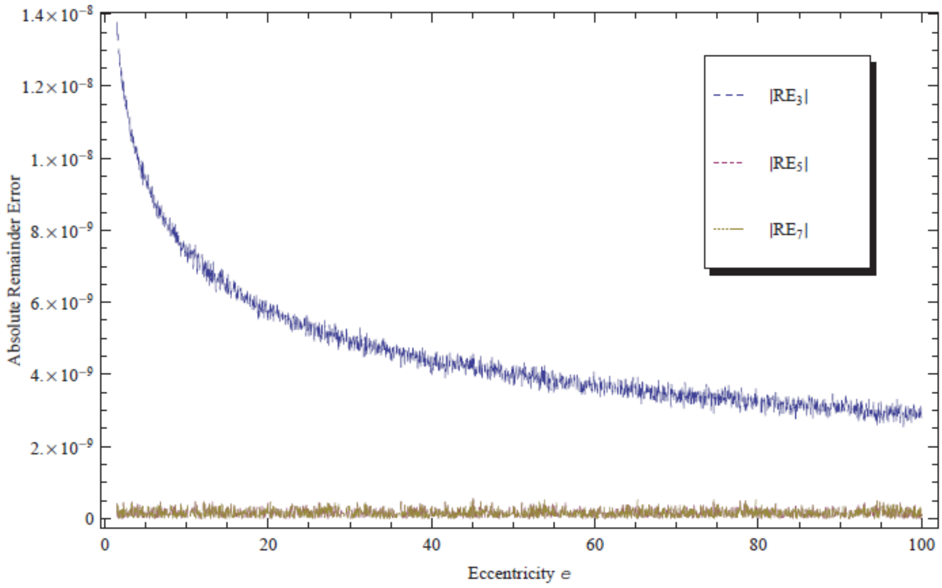

Figure 10: The absolute remainder error versus *e* at *M* = 175000.

For a further validation of the current numerical results, two additional plots are displayed in Figs. 9-10 for the absolute remainder errors $|RE_3|$, $|RE_5|$ and $|RE_7|$ versus the eccentricity e in the domain $e \in [1.5,100]$ for $M = 500$ and $M = 175000$, respectively. The results plotted in these two figures reveal that the approximate solution using only three terms of the asymptotic Adomian's series is also accurate as $M \to \infty$ at all considered values of the eccentricity parameter e. Moreover, the absolute remainder errors $|RE_5|$ and $|RE_7|$ approach zero even at these higher values of the mean anomaly and the eccentricity. This, of course, proves the several remarkable advantages of the Adomian's asymptotic decomposition method over the existing methods in the literature. Finally, the authors of the present paper recommend the current approach as the most effective analytical technique to solve the hyperbolic Kepler equation.

4. CONCLUSION

In this paper, Kepler's equation for hyperbloic orbits has been analytically solved by using the Adomian's asymptotic decomposition method (AADM). The odd property of the obtained approximate solutions has been demonstrated through a series of graphs. Also, we have shown that only a few terms of the AADM are sufficient to obtain extremely accurate numerical results even at higher values than those in the literature for the mean anomaly and the eccentricity parameters. In comparison to the methods reported in the literature, the present approach is not only effective but also very simple in technique. Moreover, the advantages of the current results for the eccentric anomaly are valid in any specified domain. Besides, very small absolute remainder errors have been achieved using only a few terms of the Adomian's asymptotic decomposition series. The effectiveness proved in this paper for Adomian's asymptotic method to quickly and easily solve the hyperbolic Kepler equation may be advantageously extended to other problems in physics and engineering.

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