

PT-SYMMETRIC OPTICAL MODES AND SPONTANEOUS SYMMETRY BREAKING IN THE SPACE-FRACTIONAL SCHRÖDINGER EQUATION

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Abstract. We numerically investigate the parity-time (PT) symmetric optical modes in the space-fractional Schrödinger equation. The PT-symmetric optical modes are found below the critical phase-transition points. Beyond the critical phase-transition points, the spontaneous symmetry breaking of the PT-symmetric system occurs for non-PT-symmetric solutions. The diagrams of the eigenvalue spectra of the optical modes are presented as function of the degree of non-Hermiticity of the system. The numerical results reveal that the critical phase-transition point increases monotonously with the increase of the Lévy index.

Key words: Space-fractional Schrödinger equation, PT-symmetry, spontaneous symmetry breaking.

1. INTRODUCTION

The fractional Schrödinger equation is a natural generalization of the non-fractional Schrödinger equation [1]. It has been introduced when the quantum phenomena were exploited [2], where the Feynman path integral over the Brownian trajectories leads to the standard Schrödinger equation. Analogously, the Feynman path integral over the Lévy trajectories leads to the fractional Schrödinger equation [3]. However, the implications of such theories are still a matter of debate [4, 5]. Therefore, most of studies on the fractional Schrödinger equation have mainly focused on the mathematical aspects, including some solutions in linear potentials [6], tunneling effects through delta and double delta potentials [7], particles moving in potential wells [8], spectral problems in finite and ultimately infinite Cauchy wells [9], and transmission through locally periodic potentials [10]; see also the comprehensive books on fractional integral transforms, fractional derivatives, and fractional dynamics [11–13].

Recently, some of experimental schemes, which aim to realize the fractional Schrödinger equation physical setting, have been proposed. In a condensed-matter environment, the one-dimensional Lévy crystal, as a probable candidate for an ex-

perimentally accessible realization of space-fractional quantum mechanics, has been introduced by Stickler [14]. An optical realization of the fractional Schrödinger equation model, based on transverse light dynamics in aspherical optical cavities, was advanced by Longhi [15], which provides a new interesting way to control the diffraction of light. The propagation of light by using dynamical models based on the fractional Schrödinger equation has been extensively investigated and a number of intriguing properties have also been reported. The wave dynamics has been studied in one- and two-dimensional models with harmonic potential, and the obtained results show that the one-dimensional beam propagates along a zigzag trajectory in real space. However, the two-dimensional beam evolves into a breathing ring structure in both real and momentum spaces [16]. Diffraction-free beams, including finite energy Airy beams and ring Airy beams, exhibit beam splitting, and the propagation trajectories of the beams are greatly influenced by the introduction of the Lévy index into the theoretical model [17–21]. Super-Gaussian beams split into two sub-beams in linear fractional Schrödinger equation, and they can evolve to become a single soliton or a soliton pair [22]. A periodically oscillating behavior of Gaussian beams has been found in the linear fractional Schrödinger equation with variable coefficients [23]. Also, beam propagation management [24], optical Bloch oscillations and Zener tunneling [25], resonant mode conversions and Rabi oscillations [26] have been investigated in the linear regime.

The other promising extension of standard Schrödinger equation is in the area of PT-symmetric systems. These systems are non-Hermitian due to the presence of gain and loss, which are exactly balanced. The key properties, investigated originally by Bender and Boettcher [27–29], are that there can exist all real eigenvalue spectra in such PT-symmetric systems. The concept of PT-symmetry, gone far beyond the quantum physics, and has been spread out to optics and photonics, Bose-Einstein condensates, plasmonic waveguides and meta-materials, superconductivity, etc.; see a few review papers and the references therein [30–36].

Recently, PT-symmetric potentials have been introduced in the models of the fractional Schrödinger equation, where the input beams propagate in the form of conical diffraction at a critical point [37]. In addition, defect modes have been found in defective PT-symmetric potentials [38]. Naturally, the fractional Schrödinger equation has been applied to nonlinear optical systems [39]. And recent progress shows that a variety of fractional optical solitons can exist in nonlinear fractional Schrödinger equation [40], examples include: accessible solitons [41–43], double-hump solitons and fundamental solitons in PT-symmetric potentials [44, 45], gap solitons and surface gap solitons in PT-symmetric optical lattices [46, 47], two-dimensional solitons [48], and off-site and on-site vortex solitons in PT-symmetric optical lattices [49]. Moreover, the relation between nonlinear Bloch waves and gap solitons has also been discussed in nonlinear fractional Schrödinger equation with

optical lattices [50]. The readers are also referred to a series of recent works in the broad area of PT-symmetric systems [51–59].

In this paper, we investigate the space-fractional Schrödinger equation with a PT-symmetric complex-valued external potential. We focus on the PT-symmetric optical modes and spontaneous symmetry breaking of the PT-symmetric system in the linear space-fractional Schrödinger equation. Both PT-symmetric and non-PT-symmetric solutions will be shown. The critical phase-transition points, belonging to different optical modes, are found. Furthermore, the diagrams of eigenvalues and the critical phase-transition points, which depend on the numerical value of the Lévy index, will be revealed.

2. THE MODEL AND ITS REDUCTION

The theoretical model for describing optical beam propagation in the framework of the space-fractional Schrödinger equation with a PT-symmetric external potential is governed by the following equation

$$i \frac{\partial A}{\partial z} - \frac{1}{2k} \left(-\frac{\partial^2}{\partial x^2} \right)^{\alpha/2} A + \frac{k[F(x) - n_0]}{n_0} A = 0, \quad (1)$$

where $A(z, x)$ is the optical field envelope function, $k = 2\pi n_0/\lambda$ is the wavenumber with λ and n_0 being the wavelength of the optical source and the background refractive index, respectively. Here α is the Lévy index ($1 < \alpha \leq 2$) and $F(x) = F_R(x) + iF_I(x)$ is a complex-valued function, in which the real part represents the linear refractive index distribution and the imaginary part stands for gain and loss. By introducing the transformations $\Psi(\zeta, \xi) = A(z, x)/A_0$, where A_0 is the input peak amplitude, $\xi = x/x_0$ is the normalized transverse coordinate, and $\zeta = z/(kx_0^\alpha)$, Eq. (1) can be rewritten in a dimensionless form

$$i \frac{\partial \Psi}{\partial \zeta} - \frac{1}{2} \left(-\frac{\partial^2}{\partial \xi^2} \right)^{\alpha/2} \Psi + U(\xi) \Psi = 0. \quad (2)$$

Here, the normalized potential can be expressed as $U(\xi) \equiv V(\xi) + iW(\xi)$ with $V(\xi) = k^2 x_0^\alpha [F_R(x_0 \xi) - n_0]/n_0$ and $W(\xi) = k^2 x_0^\alpha F_I(x_0 \xi)/n_0$, which are required to be even and odd functions, respectively, for PT-symmetric nonlinear optical waveguides.

3. PT-SYMMETRIC OPTICAL MODES AND SPONTANEOUS SYMMETRY BREAKING

To analyze the PT-symmetric optical modes and spontaneous symmetric breaking of the space-fractional Schrödinger equation (2), we need to assume the stationary solutions in the form of $\Psi(\zeta, \xi) = \phi(\xi)e^{i\beta\zeta}$, where $\phi(\xi)$ is a complex-valued

function and β is the corresponding eigenvalue. Substitution into Eq. (2) yields

$$-\frac{1}{2} \left(-\frac{d^2}{d\xi^2} \right)^{\alpha/2} \phi(\xi) + U(\xi) \phi - \beta \phi(\xi) = 0. \quad (3)$$

Assuming the phase distribution of $\varphi(\xi)$, the stationary solution can also be written as $\varphi(\xi) = |\phi(\xi)| e^{i\varphi}$. The Poynting vector can be expressed as $S = |\phi|^2 \varphi_\xi$.

Equation (3) is a stationary linear space-fractional Schrödinger equation. The analytic solution of Eq. (3) cannot be derived in general; the difficulty comes from the presence of the space-fractional derivative, i.e., $(-d^2/d\xi^2)^{\alpha/2}$, which is inherently a nonlocal operator. To obtain the eigenvalues and eigenfunctions of Eq. (3), we will use the Fourier collocation method [60].

In this paper, we consider a super-Gaussian-type PT-symmetric potential in the form

$$V(\xi) = V_0 e^{-(\xi/\xi_0)^{2m}}, \quad W(\xi) = W_0 (\xi/\xi_0) e^{-(\xi/\xi_0)^{2m}}, \quad (4)$$

where the parameters V_0 and W_0 are the normalized modulation strength of the refractive index and the balanced gain and loss, in which W_0 characterizes the degree of non-Hermiticity of the PT-symmetric system, ξ_0 is the potential width, and m is the power index of super-Gaussian function. The value $m = 1$ denotes a Gaussian potential, and the profile of $V(\xi)$ gradually resembles a rectangular distribution with increasing m . Here, the power index of super-Gaussian function is $m = 2$. The modulation strength of the refractive index is $V_0 = 6$ and the potential width is $\xi_0 = 4$. The degree of non-Hermiticity varies from 0 to 8. For the case of $W_0 = 0$, the system described by Eq. (2) degenerates into a Hermitian one.

Without loss of generality, we choose the value of the Lévy index $\alpha = 1.5$. Using the Fourier collocation method, Eq. (3) can be solved numerically. The eigenvalue spectra, depending on the degree of non-Hermiticity, are shown in Fig. 1, where the real and imaginary components of the eigenvalue spectra are presented in the first row and the second row of Fig. 1, respectively. Below the first critical phase-transition point ($W_{cr1} \approx 1.293$), eight real eigenvalues, corresponding to the PT-symmetric optical modes, are found in this fractional system. The real components of the eigenvalue spectra for the 1st and 2nd modes merge in pairs with increasing W_0 , as shown in Fig. 1(a₁). Beyond the first critical phase-transition point W_{cr1} , the real eigenvalues turn into complex ones, the imaginary components are shown in Fig. 1(b₁). It is indicated that the spontaneous symmetry breaking of the PT-symmetric system occurs when the 1st and 2nd PT-symmetric modes become non-PT-symmetric ones. However, the rest of modes still preserve the PT-symmetry until W_0 increases approximately to 2.343, when the eigenvalue spectra of the 3rd and 4th modes merge into the second critical phase-transition point W_{cr2} as shown in Fig. 1(a₂). Then, the PT-symmetry of the 3rd and 4th modes is broken, and the corresponding eigenvalues become complex as shown in Fig. 1(b₂). The third and fourth

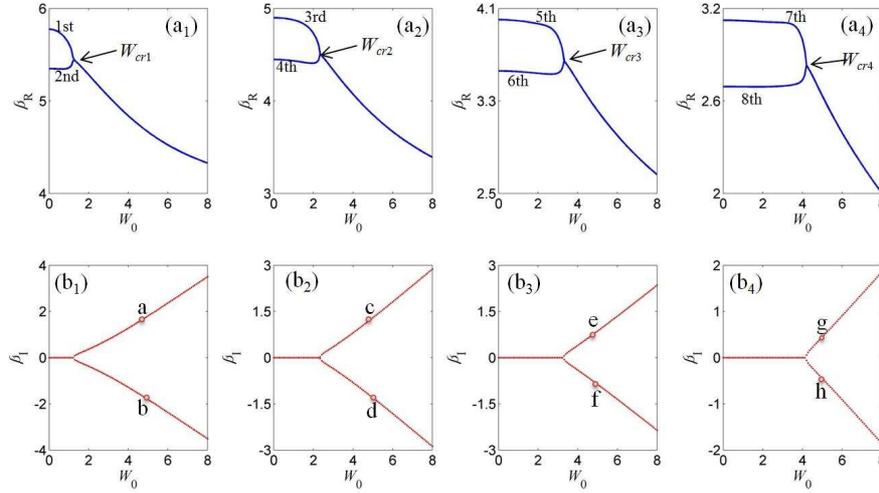


Fig. 1 – Eigenvalue spectra depend on the degree of non-Hermiticity. (a₁) and (b₁) show the real part (blue curves) and imaginary part (red curves) of the eigenvalue spectrum for the 1st (fundamental) and 2nd modes. (a₂) and (b₂) show the real and imaginary parts of the eigenvalue spectrum for the 3rd and 4th modes. (a₃) and (b₃) show the real and imaginary parts of the eigenvalue spectrum for the 5th and 6th modes. (a₄) and (b₄) show the real and imaginary parts of the eigenvalue spectrum for the 7th and 8th modes. Here, the power index of the super-Gaussian potential is $m = 2$ and the Lévy index equals 1.5.

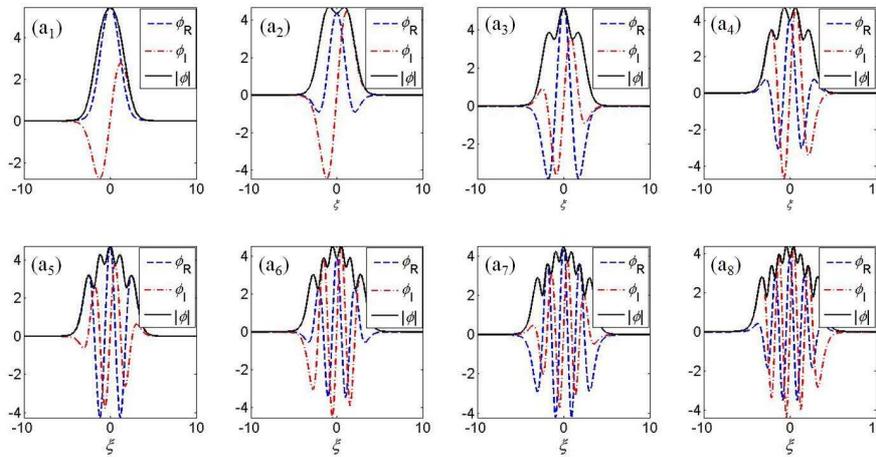


Fig. 2 – PT-symmetric modes with real eigenvalues. In (a₁) and (a₂) we show the 1st (fundamental) and the 2nd modes. Here we display the real part (blue dashed), the imaginary part (red dash-dot), and the spatial distribution (black solid) of the modes. (a₃) and (a₄) show the 3rd and 4th modes, (a₅) and (a₆) show the 5th and 6th modes, and (a₇) and (a₈) show the 7th and 8th modes. The degree of non-Hermiticity $W_0 = 1$ and the other parameters are the same as in the Fig. 1.

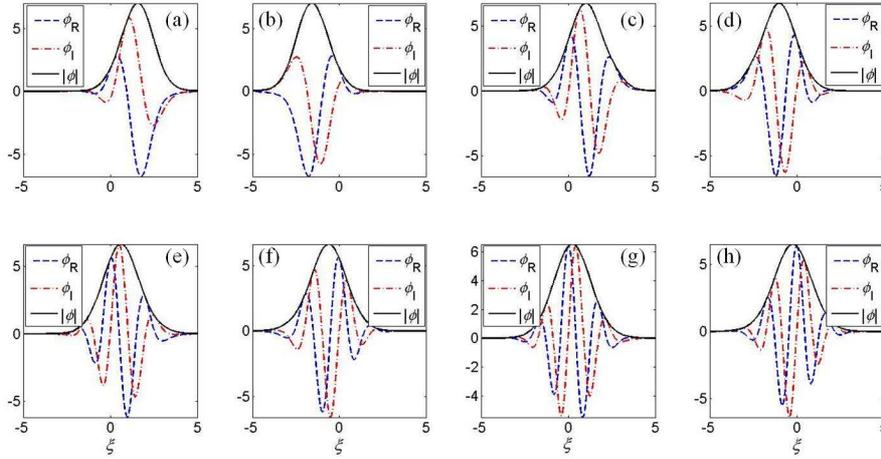


Fig. 3 – Non-PT-symmetric solutions with complex eigenvalues. (a) and (b) depict the symmetry breaking of the optical modes marked with “a” and “b” in Fig. 1(b₁). Here we show the real part (blue dashed), the imaginary part (red dash-dot), and the spatial distribution (black solid) of the modes. The panels (c)–(h) show the non-PT-symmetric modes corresponding to the points of “c”, “d”, “e”, “f”, “g”, and “h” marked with red circles in Fig. 1(b₂), (b₃), and (b₄), respectively. The degree of non-Hermiticity is $W_0 = 5$ and the other parameters are the same as in Fig. 1.

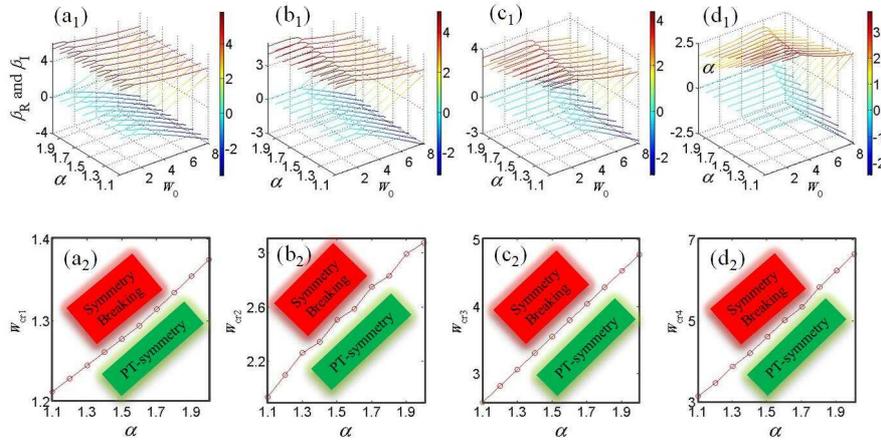


Fig. 4 – Three-dimensional diagrams of eigenvalues vs. the Lévy index and the degree of non-Hermiticity. Panels (a₁), (a₂), (a₃), and (a₄) depict the dependence of the real and imaginary components of the pair-merged eigenvalue spectra (i.e., 1st and 2nd, 3rd and 4th, 5th and 6th, 7th and 8th) on the Lévy index ($1.1 \leq \alpha \leq 2$) and the degree of non-Hermiticity ($0 \leq W_0 \leq 8$). Panels (b₁), (b₂), (b₃), and (b₄) show the dependence of the critical phase-transition points on the numerical value of the Lévy index α . The parameters of the PT-symmetric potential are the same as in Fig. 1.

critical phase-transition points ($W_{cr3} \approx 3.313$, $W_{cr4} \approx 4.202$) emerge sequentially with increasing W_0 as shown in Fig. 1(a₃) and Fig. 1(a₄), respectively; the corresponding imaginary components of the complex eigenvalues are depicted in Fig. 1(b₃) and Fig. 1(b₄).

The modes of Eq. (3) with Lévy index $\alpha = 1.5$, similar to the modes of the standard Schrödinger equation with a PT-symmetric potential, satisfy the PT-symmetry and are depicted in Fig. 2(a₁)-(a₈). The real and imaginary components of the fundamental (1st) mode display the even and odd symmetry, respectively, see Fig. 2(a₁). The symmetries of the real and imaginary profiles are alternately changing between odd and even ones for the high-order optical modes as shown in Fig. 2(a₂)-(a₈). The width of the profiles for the high-order optical modes increases with the order of the PT-symmetric mode, in order to maintain the balance of transverse energy. Thus, the high-order optical modes can still keep the PT-symmetry at a large degree of non-Hermiticity (i.e., $W_{cr1} < W_{cr2} < W_{cr3} < W_{cr4}$). Non-PT-symmetric solutions with complex eigenvalues are presented in Fig. 3, where the real and imaginary profiles of the solution are asymmetric. We see that the balance of transverse energy is spoiled, so that the PT-symmetric solution cannot exist. Due to the presence of the imaginary parts of the complex eigenvalues, the amplitudes of the non-PT-symmetric solutions will be exponentially amplified or attenuated during propagation, according to the signs of imaginary components.

To examine the dependence of the critical phase-transition points on Lévy index, Eq. (3) is numerically solved for different values of the Lévy index α and of the modulation strengths of the degree of non-Hermiticity W_0 . The eigenvalue spectra are depicted in Fig. 4 with different values of the Lévy index α ranging from 1.1 to 2. From the bottom row of Fig. 4, it is found that the critical phase-transition point increases monotonously with increasing the Lévy index. And the domains of the parameters (α - W_{cr}), bounded by the critical phase-transition points, are divided into two parts, namely the PT-symmetric phase and the symmetry-broken phase.

4. CONCLUSIONS

In this paper, we have investigated the PT-symmetric optical modes in the space-fractional Schrödinger equation. The PT-symmetric optical modes and non-PT-symmetric solutions have been found below and beyond the critical phase-transition points, respectively. The spontaneous symmetry breaking of the PT-symmetric system has been revealed for different values of the Lévy index. Finally, the dependence of the critical phase-transition point on the numerical value of Lévy index has been investigated. The results show that the critical phase-transition point increases

monotonously with the increase of the Lévy index.

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