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RELIABLE TREATMENT FOR SOLVING BOUNDARY VALUE PROBLEMS OF PANTOGRAPH DELAY DIFFERENTIAL EQUATION

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Abstract. This work presents an accurate and reliable treatment of the pantograph equation, which is a delay differential equation that appears in many scientific applications. The Adomian decomposition method and the variational iteration method will be used to carry out this work. Both the Adomian decomposition method and the variational iteration method provide convergent series solutions for linear and nonlinear differential equations. We conduct a comparative study between the two methods by highlighting the specific features of each method. Four linear and nonlinear pantograph equations will be investigated to support this work. The power of the two methods is confirmed.

Key words: Adomian decomposition method; variational iteration method; pantograph equation; delay differential equation.

1. INTRODUCTION

The pantograph equations, a kind of delay differential equations, arise in many applications such as electrodynamics, astrophysics, nonlinear dynamical systems, biology, control problems, probability theory on algebraic structures, quantum mechanics, cell growth etc. [1–16]; see also the recent works [17–22] for other applications of either ordinary differential equations or partial differential equations in pure and applied sciences and engineering. Usually ordinary differential equations involve derivatives that depend on the solution at the present time. However, delay differential equations contain in addition derivatives that not only depend on the present state but also depend on past times. A pantograph is a device that collects electronic current from overhead lines for electric trains or trams. The name pantograph originated from the work of Ockendon and Taylor [8] on the collection of current by the pantograph head of an electric locomotive.

The pantograph equations have been studied by many authors, who have investigated both analytical and numerical aspects [2]. A variety of analytical and numerical schemes, such as the Runge–Kutta method, the collocation method, the homotopy analysis method, extrapolation schemes, shooting methods, and other methods were used to examine the properties of the solutions of the pantograph equation.

We aim in this work to conduct a comparative study between the Adomian decomposition method (ADM) [1, 9–16] and the variational iteration method (VIM) [6] for solving the second-order pantograph delay differential equations of the form

$$\begin{aligned} u''(x) &= g(x)u'(x)+h(x)u(qx)+f(x), 0<x<X, \\ u(0) &= a, u(1)=b, \end{aligned} \tag{1}$$

where $f(x)$, $g(x)$, and $h(x)$ are analytic functions, $0<q<1$, and a and b are constants.

The pantograph equation attracted many research efforts during the past years. Chen and Wang [3] applied the variational iteration method for solving a neutral functional–differential equation with proportional delays, where high-accuracy approximate solutions were achieved after only a few iterations. A new Jacobi rational–Gauss collocation method for numerical solution of generalized pantograph equations was employed by Doha *et al.* [4].

Evans and Raslan [5] used the Adomian decomposition method to handle the delay differential equations where appropriate approximations were obtained. Recently, generalized pantograph equations were handled by Javadi *et al.* [7] by using shifted orthonormal Bernstein polynomials, where the pantograph equation was converted to a system of linear equations. Other methods used a direct solution technique for solving the generalized pantograph equation with variable coefficients subject to initial conditions, where a collocation method based on Bernoulli operational matrix of derivatives was used. Other researchers used other reliable algorithms on solving the pantograph equation.

We point out that the first order pantograph equation was examined heavily in the literature. The aforementioned ADM and the VIM methods have found their ways into many different fields of differential and integral equations for numerical and analytical purposes. The two methods have gained considerable attention in solving scientific and engineering models. As stated before we will use the ADM and the VIM methods to obtain the convergent series solutions. Moreover, a comparative study will be conducted to highlight the features and the power of each method.

2. ANALYSIS OF THE METHODS

The ADM and the VIM are well documented in the literature. However, we will outline the necessary steps of each method and allows the reader to follow our analysis in the sequel.

2.1. The Adomian decomposition method

We first write down the pantograph delay differential equation, a boundary value problem, in a general fashion as

$$u''(x) = g(x)u'(x) + h(x)u(qx) + f(x),\ 0 \leq x \leq 1,\ 0 < q < 1, \tag{2}$$

with two point boundary value conditions given by

$$u(0) = a,\, u(1) = b. \tag{3}$$

We rewrite the generalized pantograph equation in Adomians operator-theoretic form

$$Lu + Nu = 0, \tag{4}$$

where

$$\begin{aligned} Lu &= u''(x), \\ Nu &= g(x)u'(x) + h(x)u(qx) + f(x), \end{aligned} \tag{5}$$

and consequently, the inverse linear operator L^{-1} is defined by

$$L^{-1}(.) = \int_0^x \int_0^x (.)\, dx\, dx. \tag{6}$$

Applying the inverse linear operator L^{-1} to both sides of (2) gives

$$u(x) = \alpha + \beta x + r(x) + L^{-1}\left(g(x)u'(x) + h(x)u(qx)\right), \tag{7}$$

where $r(x) = L^{-1}(f(x))$, $\alpha = u(0) = a$ and $\beta = u'(0)$ that will be determined later by using the other boundary condition.

The Adomian decomposition method admits the use of the infinite decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{8}$$

for the solution $u(x)$, and the infinite series of polynomials

$$F(u) = \sum_{n=0}^{\infty} A_n(y_0, y_1, \cdots, y_n), \tag{9}$$

for the nonlinear term $F(u)$, where the components $u_n(x)$ of the solution $u(x)$ will be determined recurrently, and A_n are the Adomian polynomials that can be constructed according to a variety of algorithms given in Refs. [1, 9–16].

Substituting (8) and (9) into (7) yields

$$\sum_{n=0}^{\infty} u_n(x) = \alpha + \beta x + r(x) + L^{-1}\left(g(x)\left(\sum_{n=0}^{\infty} u_n(x)\right)' + h(x)\left(\sum_{n=0}^{\infty} u_n(qx)\right)\right). \tag{10}$$

Identifying $u_0(x) = \alpha + \beta x + r(x)$, the recursive relation

$$\begin{aligned} u_0(x) &= \alpha + \beta x + r(x), \\ u_{k+1}(x) &= L^{-1}\left(g(x)\,(u_k(x))' + h(x)\,(u_k(qx))\right), k \geq 0, \end{aligned} \tag{11}$$

will lead to the complete determination of the components $u_n(x)$ of $u(x)$.

The series solution of $u(x)$ follows immediately. The series solution will converge to the exact solution if such a solution exists. However, for concrete problems where exact solution is not obtainable, the obtained series solution can be used for numerical purposes.

2.2. The VIM and the Lagrange multipliers

Next we will outline the necessary steps for using the variational iteration method which requires the use of the Lagrange multipliers, which may be a constant or a variable depending on the equation itself. Consider the differential equation

$$Lu + Nu = 0, \tag{12}$$

where L and N are linear and nonlinear operators respectively, and $f(x)$ is the source term.

To use the VIM, a correction functional for equation (12) should be used in the form

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\,(Lu_n(\xi) + N\,\tilde{u}_n(\xi))\,d\xi, \tag{13}$$

where λ is a general Lagrange's multiplier, which can be identified optimally *via* the variational theory, and $\tilde{u}_n$ as a restricted variation which means $\delta\tilde{u}_n = 0$.

For Eqs. (2), the correction functional reads

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\,((u_n(\xi))_{\xi\xi} + g(\xi)\,(u_n(\xi))_\xi + h(\xi)\tilde{u}_n(q\xi))\,d\xi, \tag{14}$$

where $\delta(\tilde{u}_n(q\xi) = 0$.

To determine the optimal value of $\lambda(\xi)$, we take the variation for both sides with respect to $u_n(x)$ to obtain

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta \int_0^x \lambda(\xi)\,((u_n(\xi))_{\xi\xi} + g(\xi)\,(u_n(\xi))_\xi)\,d\xi, \tag{15}$$

where we used $\delta(\tilde{u}_n(q\xi) = 0$. Integrating the integral on the right side by parts we get the stationary conditions

$$\begin{aligned} \lambda|_{\xi=x} &= 0, \\ 1 - \lambda'|_{\xi=x} &= 0, \\ \lambda''|_{\xi=x} &= 0. \end{aligned} \tag{16}$$

This in turn gives $\lambda = \xi - x$.

The successive approximations $u_{n+1}, n \geq 0$ of the solution $u(x)$ will be readily obtained upon using any selective function $u_0(x)$. Consequently, we get the solution

$$u(x) = \lim_{n\to\infty} u_n(x). \tag{17}$$

It is interesting to note that the variational iteration method gives several approximations, and therefore the exact solution is obtained at the limit of the resulting successive approximations. However, the Adomian decomposition method gives components of the solution, where by adding these components we get a series of these components.

In what follows, we will examine four numerical examples of pantograph delay differential equations that were examined in [8] by using computational intelligent algorithms. These equations were first introduced in [8] and were examined by the method of successive interpolations.

3. NUMERICAL EXAMPLES

In this Section, we will examine four numerical examples of pantograph delay differential equations that were examined in [13] by using computational intelligent algorithms. These equations were first introduced in [13] and were examined the method of successive interpolations. The treatments in [13] resulted in approximation of high accuracy.

3.1. Problem 1

We first study the nonhomogeneous two-point boundary value problem of second order pantograph delay differential equation

$$u''(x) = \frac{1}{2}u + e^{-\frac{x}{2}}u\left(\frac{1}{2}x\right) - 2e^{-x}, x \in [0,1], \tag{18}$$

with boundary conditions

$$u(0) = 0, u(1) = e^{-1}. \tag{19}$$

3.1.1. Using the ADM

To solve the pantograph equation (18) by using the Adomian decomposition method, we apply first the inverse integral operator L^{-1} to both sides, then we set the recurrence relation

$$\begin{array}{rcl} u_0(x) &=& \beta x - (2e^{-x} + 2x - 2), \\ u_{k+1}(x) &=& \int_0^x \int_0^t \left(\frac{1}{2}u_k(x) + e^{-x}u_k(\frac{1}{2}x)\right) dx\, dt,\ k \geq 0, \end{array} \tag{20}$$

where $\beta = u'(0)$ that will be determined later. The zeroth component was assigned by using the initial conditions and by integrating $-2e^{-x}$ twice.

Using (20) we obtain the following approximations

$$\begin{aligned}
u_0(x) &= \beta x - (2e^{-x} + 2x - 2),\\
u_1(x) &= \tfrac{\beta}{6}x^3 - (\tfrac{\beta}{48} + \tfrac{1}{16})x^4 + (\tfrac{\beta}{320} + \tfrac{1}{60})x^5 - (\tfrac{\beta}{2880} + \tfrac{19}{5760})x^6 + (\tfrac{\beta}{32256} + \tfrac{1}{1920})x^7\\
&- (\tfrac{1}{430080}\beta + \tfrac{89}{1290240})x^8 + (\tfrac{1}{6635520}\beta + \tfrac{23}{2903040})x^9\\
&- (\tfrac{1}{116121600}\beta + \tfrac{5}{6193152})x^{10} + O(x^{11}),\\
u_2(x) &= \tfrac{\beta}{192}x^5 - (\tfrac{17\beta}{23040} + \tfrac{3}{2560})x^6 + (\tfrac{151}{1290240}\beta + \tfrac{83}{322560})x^7\\
&- (\tfrac{19}{1290240}\beta - \tfrac{43}{983040})x^8 + (\tfrac{161}{26542080} + \tfrac{257}{165150720}\beta)x^9\\
&- (\tfrac{121}{849346560}\beta + \tfrac{283}{396361728})x^{10} + O(x^{11}),\\
u_3(x) &= \tfrac{17}{258048}\beta x^7 - (\tfrac{227}{27525120}\beta + \tfrac{99}{9175040}x^8) + (\tfrac{4709}{3963617280}\beta + \tfrac{5773}{2972712960})x^9\\
&- (\tfrac{2099}{14863564800}\beta + \tfrac{26779}{95126814720})x^{10} + O(x^{11}),\\
&\vdots
\end{aligned} \tag{21}$$

The series solution is therefore given by

$$\begin{aligned}
u(x) &= \beta x - x^2 + (\tfrac{\beta}{6} + \tfrac{1}{3})x^3 - (\tfrac{\beta}{48} + \tfrac{7}{48})x^4 + (\tfrac{\beta}{120} + \tfrac{1}{30})x^5 - (\tfrac{5\beta}{4608} + \tfrac{167}{23040})x^6\\
&+ (\tfrac{23\beta}{107520} + \tfrac{379}{322560})x^7 - (\tfrac{953}{5505024} + \tfrac{2089}{82575360}\beta)x^8\\
&+ (\tfrac{21247}{990904320} + \tfrac{34423}{11890851840}\beta)x^9 - (\tfrac{373213}{158544691200} + \tfrac{8689}{29727129600}\beta)x^{10}\\
&+ O(x^{11}).
\end{aligned} \tag{22}$$

To determine β, and hence to obtain the exact solution, we use the approximants

$$\phi_i = \sum_{i=0}^{i=3} u_i(x). \tag{23}$$

By substituting the boundary condition $u(1) = e^{-1}$ into the two-term ϕ_1, three-term ϕ_2, and four-term ϕ_3 approximants, and solving the resulting equations we obtain the following sequence of values for β given by

$$1.003196215,\ 1.000043330,\ 1.000000137, \cdots, \tag{24}$$

and hence this sequences converges to $\beta = 1$.

Substituting this value of $\beta = 1$ into (22) leads to the series solutions

$$u(x) = x\left(1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5 + \frac{1}{6!}x^6 - \frac{1}{7!}x^7 + \cdots\right), \tag{25}$$

that leads to the exact solution

$$u(x) = xe^{-x}. \tag{26}$$

3.1.2. Using the VIM

The correction functional for (18) reads

$$u_{n+1}(x) = u_n(x) + \int_0^x (t-x)\left(u_n''(t) + 2e^{-t} - \frac{1}{2}u_n(t) - e^{-\frac{1}{2}t}u_n(\frac{1}{2}t)\right)dt,\ n \geq 0, \tag{27}$$

where we used $\lambda = t - x$ as given above.

Considering the given initial values, we can select $y_0(x) = 0 + \beta x$, where $\beta = u'(0)$. Using this selection into (27) we obtain the following successive approximations

$$\begin{aligned}
u_0(x) &= \beta x,\\
u_1(x) &= \beta x - x^2 + (\tfrac{\beta}{6} + \tfrac{1}{3})x^3 - (\tfrac{\beta}{48} + \tfrac{1}{12})x^4 + (\tfrac{\beta}{320} + \tfrac{1}{60})x^5 - (\tfrac{\beta}{2880} + \tfrac{1}{360})x^6 +\\
&+ (\tfrac{\beta}{32256} + \tfrac{1}{2520})x^7 - (\tfrac{1}{20160} + \tfrac{1}{430080}\beta)x^8 + (\tfrac{1}{181440} + \tfrac{1}{6635520}\beta)x^9\\
&- (\tfrac{1}{1814400} + \tfrac{1}{116121600}\beta)x^{10} + O(x^{11}),\\
u_2(x) &= \beta x - x^2 + (\tfrac{\beta}{6} + \tfrac{1}{3})x^3 - (\tfrac{\beta}{48} + \tfrac{7}{48})x^4 + (\tfrac{\beta}{120} + \tfrac{1}{30})x^5 - (\tfrac{5\beta}{4608} + \tfrac{7}{1152})x^6\\
&+ (\tfrac{191\beta}{1290240} + \tfrac{37}{40320})x^7 - (\tfrac{17}{143360} + \tfrac{11}{645120}\beta)x^8\\
&+ (\tfrac{13}{967680} + \tfrac{2537}{1486356480}\beta)x^9 - (\tfrac{631}{464486400} + \tfrac{499}{3303014400}\beta)x^{10} + O(x^{11}),\\
u_3(x) &= \beta x - x^2 + (\tfrac{\beta}{6} + \tfrac{1}{3})x^3 - (\tfrac{\beta}{48} + \tfrac{7}{48})x^4 + (\tfrac{\beta}{120} + \tfrac{1}{30})x^5 - (\tfrac{5\beta}{4608} + \tfrac{167}{23040})x^6\\
&+ (\tfrac{23\beta}{107520} + \tfrac{379}{322560})x^7 - (\tfrac{1117}{6881280} + \tfrac{2089}{82575360}\beta)x^8\\
&+ (\tfrac{3623}{185794560} + \tfrac{34423}{11890851840}\beta)x^9 - (\tfrac{61609}{29727129600} + \tfrac{8689}{29727129600}\beta)x^{10}\\
&+ O(x^{11}),\\
&\vdots\quad .
\end{aligned} \tag{28}$$

To determine β, and hence to obtain the exact solution, we substitute the boundary condition $u(1) = e^{-1}$ into the approximations u_1, u_2, and u_3, and solving the resulting equations we obtain the following sequence of values for β given by

$$0.9607811818, 0.9991113707, 0.9999472947, \cdots, \tag{29}$$

and as a result, this sequences converges to $\beta = 1$.

Substituting this value of $\beta = 1$ into $u_3(x)$ in (28) leads to the series solutions

$$u(x) = x\left(1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5 + \frac{1}{6!}x^6 - \frac{1}{7!}x^7 + \cdots\right), \tag{30}$$

that leads to the exact solution

$$u(x) = xe^{-x}. \tag{31}$$

3.2. Problem 2

We next study the nonhomogeneous two-point boundary value problem of second order pantograph delay differential equation:

$$u''(x) = 1+2\left(1+\frac{1}{8}x^2\right)\cos\left(\frac{1}{2}x\right)-2\cos\left(\frac{1}{2}x\right)u\left(\frac{1}{2}x\right),\ x\in[0,\frac{\pi}{4}], \tag{32}$$

subject to the boundary conditions

$$u(0)=1,\ u\left(\frac{\pi}{4}\right)=1+\frac{\sqrt{2}}{2}+\frac{\pi^2}{32}. \tag{33}$$

3.2.1. Using the ADM

To solve the pantograph equation (32) by using the Adomian decomposition method, we apply first the inverse integral operator L^{-1} to both sides, then we set the recurrence relation

$$\begin{array}{rcl} u_0(x) &=& -15+\beta x+\frac{1}{2}x^2+(16-x^2)\cos(\frac{1}{2}x)+8x\sin(\frac{1}{2}x), \\ u_{k+1}(x) &=& \int_0^x\int_0^t\left(-2\cos(\frac{1}{2}x)u_k(\frac{1}{2}x)\right)dx\,dt, k\geq 0, \end{array} \tag{34}$$

where $\beta = u'(0)$ that will be determined later. The zeroth component was assigned by using the initial conditions and by integrating $1+2(1+\frac{1}{8}x^2)\cos(\frac{1}{2}x)$ twice.

Using (34) we obtain the following approximations

$$\begin{array}{rcl} u_0(x) &=& -15+\beta x+\frac{1}{2}x^2+(16-x^2)\cos(\frac{1}{2}x)+8x\sin(\frac{1}{2}x), \\ u_1(x) &=& -x^2-\frac{\beta}{6}x^3-\frac{1}{24}x^4+\frac{\beta}{160}x^5+\frac{17}{5760}x^6-\frac{\beta}{16128}x^7+O(x^8), \\ u_2(x) &=& \frac{1}{24}x^4+\frac{\beta}{480}x^5-\frac{11}{5760}x^6-\frac{43\beta}{322560}x^7+O(x^8), \\ u_3(x) &=& -\frac{1}{5760}x^6-\frac{\beta}{322560}x^7+O(x^8), \\ \vdots && \end{array} \tag{35}$$

The series solution is therefore given by

$$u(x)=1+\beta x+\frac{1}{2}x^2-\frac{\beta}{6}x^3+\frac{\beta}{120}x^5-\frac{\beta}{5040}x^7+O(x^8) \tag{36}$$

To determine β, and hence to obtain the exact solution, we use the approximants

$$\phi_i=\sum_{i=0}^{i=3}u_i(x). \tag{37}$$

By substituting the boundary condition $u(\frac{\pi}{4})=1+\frac{\sqrt{2}}{2}+\frac{\pi^2}{32}$ into the two-term ϕ_1, three-term ϕ_2, and four-term ϕ_3 approximants, and solving the resulting equations we obtain the following sequence of values for β given by

$$1.022598967,\ 1.0005322,\ 1.000015672,\cdots, \tag{38}$$

where this sequences converges to $\beta = 1$. Substituting this value of $\beta = 1$ into (36) leads to the series solutions

$$y(x) = 1 + \frac{1}{2}x^2 + (x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots, \tag{39}$$

that leads to the exact solution

$$u(x) = 1 + \frac{1}{2}x^2 + \sin x. \tag{40}$$

3.2.2. Using the VIM

The correction functional for (32) reads

$$\begin{array}{l} u_{n+1}(x) = u_n(x) \\ + \int_0^x (t-x)\left(u_n''(t) - 1 - 2(1+\frac{1}{8}t^2)\cos(\frac{1}{2}t) - 2\cos(\frac{1}{2}t)u_n(\frac{1}{2}t)\right) dt, n \geq 0, \end{array} \tag{41}$$

where we used $\lambda = t - x$ as given above.

Considering the given initial values, we can select $u_0(x) = 1 + \beta x$, where $\beta = u'(0)$. Using this selection into (41) we obtain the following successive approximations

$$\begin{array}{l} u_0(x) = 1 + \beta x, \\ u_1(x) = 1 + \beta x + \frac{1}{2}x^2 - \frac{\beta}{6}x^3 + \frac{1}{48}x^4 + \frac{\beta}{160}x^5 - \frac{1}{960}x^6 - \frac{\beta}{16128}x^7 + O(x^8) \\ u_2(2) = 1 + \beta x + \frac{1}{2}x^2 - \frac{\beta}{6}x^3 + \frac{\beta}{120}x^5 - \frac{1}{11520}x^6 - \frac{\beta}{5120}x^7 + O(x^8) \\ u_3(x) = 1 + \beta x + \frac{1}{2}x^2 - \frac{\beta}{6}x^3 + \frac{\beta}{120}x^5 - \frac{\beta}{5040}x^7 + O(x^8) \\ \vdots \end{array} \tag{42}$$

To determine β, and hence to obtain the exact solution, we substitute the boundary condition $u(\frac{\pi}{4}) = 1 + \frac{\sqrt{2}}{2} + \frac{\pi^2}{32}$ into the approximations $u_1, u_2,$, and u_3, and solving the resulting equations we obtain the following sequence of values for β given by

$$0.9899719405, 1.000028446, 1.000000440, \cdots, \tag{43}$$

and proceeding as before, we find $\beta = 1$. Substituting this value of $\beta = 1$ into (36) leads to the the exact solution

$$u(x) = 1 + \frac{1}{2}x^2 + \sin x. \tag{44}$$

3.3. Problem 3

We now solve the homogeneous two-point boundary value problem of second order pantograph delay differential equation:

$$u''(x) = 4e^{-\frac{x}{4}}\sin(\frac{1}{2}x)u(\frac{1}{2}x), x \in [0, \frac{\pi}{4}], \tag{45}$$

subject to the boundary conditions

$$u(0)=1,\ u\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}e^{-\frac{\pi}{4}}. \tag{46}$$

3.3.1. Using the ADM

Proceeding as before leads to the recurrence relation

$$\begin{aligned} u_0(x) &= 1+\beta x, \\ u_{k+1}(x) &= \int_0^x \int_0^t \left(4e^{-\frac{x}{4}}\sin(\tfrac{1}{2}x)u(\tfrac{1}{2}x)\right) dx\,dt,\ k\geq 0, \end{aligned} \tag{47}$$

where $\beta = u'(0)$ that will be determined later. This in turn gives the following components

$$\begin{aligned} u_0(x) &= 1+\beta x, \\ u_1(x) &= \tfrac{1}{3}x^3+(\tfrac{\beta}{12}-\tfrac{1}{12})x^4+(\tfrac{1}{120}-\tfrac{\beta}{40})x^5+\tfrac{\beta}{360}x^6-\tfrac{1}{10080}x^7+O(x^8), \\ u_2(x) &= \tfrac{1}{360}x^6+(\tfrac{\beta}{4032}-\tfrac{5}{4032})x^7+O(x^8), \\ &\vdots \end{aligned} \tag{48}$$

The series solution is therefore given by

$$\begin{aligned} &u(x)=1+\beta x+\tfrac{1}{3}x^3+(\tfrac{\beta}{12}-\tfrac{1}{12})x^4+(\tfrac{1}{120}-\tfrac{\beta}{40})x^5+(\tfrac{\beta}{360}+\tfrac{1}{360})x^6 \\ &+(\tfrac{\beta}{4032}-\tfrac{3}{2240})x^7+O(x^8). \end{aligned} \tag{49}$$

To determine β, we substitute the boundary condition $u(\frac{\pi}{4})=\frac{\sqrt{2}}{2}e^{-\frac{\pi}{4}}$ into the two-term ϕ_1, three-term ϕ_2, and four-term ϕ_3 approximants, and solving the resulting equations we obtain the following sequence of values for β given by

$$-0.9994772566,\ -0.9999999114,\ -0.9999999611,\cdots, \tag{50}$$

where we find that this sequences converges to $\beta=-1$. Substituting this value of $\beta=1$ into (49) leads to the series solutions

$$y(x)=1-x+\frac{1}{3}x^3-\frac{1}{6}x^4+\frac{1}{30}x^5-\frac{1}{630}x^7+\cdots, \tag{51}$$

which leads to the exact solution

$$u(x)=e^{-x}\cos x. \tag{52}$$

3.3.2. Using the VIM

The correction functional for (45) reads

$$u_{n+1}(x)=u_n(x)+\int_0^x (t-x)\left(u_n''(t)-4e^{-\frac{t}{4}}\sin(\frac{1}{2}t)u_n(\frac{1}{2}t)\right) dt, n\geq 0, \tag{53}$$

where we used $\lambda = t-x$ as given earlier.

Considering the given initial values, we can select $u_0(x) = 1+\beta x$, where $\beta = u'(0)$. Using this selection into (53) we obtain the following successive approximations

$$\begin{aligned}
&u_0(x) = 1+\beta x,\\
&u_1(x) = 1+\beta x+\tfrac{1}{3}x^3+(\tfrac{\beta}{12}-\tfrac{1}{12})x^4+(\tfrac{1}{120}-\tfrac{\beta}{40})x^5+\tfrac{\beta}{360}x^6-\tfrac{1}{10080}x^7+O(x^8)\\
&u_2(2) = 1+\beta x+\tfrac{1}{3}x^3+(\tfrac{\beta}{12}-\tfrac{1}{12})x^4+(\tfrac{1}{120}-\tfrac{\beta}{40})x^5+(\tfrac{\beta}{360}+\tfrac{1}{360})x^6\\
&+(\tfrac{\beta}{4032}-\tfrac{3}{2240})x^7+O(x^8),\\
&\vdots
\end{aligned} \tag{54}$$

To determine β, and hence to obtain the exact solution, we substitute the boundary condition $u(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}e^{-\frac{\pi}{4}}$ into the approximations u_0, u_1, and u_2, and solving the resulting equations we obtain the following sequence of values for β given by

$$-0.8627510090,\ -0.9994691375,\ -0.9999352251, \cdots, \tag{55}$$

where this sequence converges to $\beta = -1$. This leads to the exact solution obtained earlier by using the Adomian decomposition method.

3.4. Problem 4

We close this work by studying the nonhomogeneous nonlinear problem given by the two-point boundary value problem of second order pantograph delay differential equation

$$u''(x) = (u^2(x)+u^3(x))u(\frac{1}{2}x), x \in [0,1], \tag{56}$$

with boundary conditions

$$u(0) = 1, u(1) = \frac{1}{2}. \tag{57}$$

3.4.1 Using the ADM

Because this is a nonlinear equation, it is normal to express the nonlinear terms $u^2(x)$ and $u^3(x)$ by Adomian polynomials A_n and B_n in the form

$$\begin{aligned}
A_0 &= u_0^2,\\
A_1 &= 2u_0u_1,\\
A_2 &= 2u_0u_2+u_1^2,\\
A_3 &= 2u_0u_3+2u_1u_2,\\
&\vdots,
\end{aligned} \tag{58}$$

and

$$\begin{array}{rcl} B_0 & = & u_0^3, \\ B_1 & = & 3u_0^2 u_1, \\ B_2 & = & 3u_0^2 u_2 + 3u_0 u_1^2, \\ B_3 & = & 3u_0^2 u_3 + 6u_0 u_1 u_2 + u_1^3, \\ \vdots, & & \end{array} \tag{59}$$

respectively.

Proceeding as before we set the recurrence relation

$$\begin{array}{rcl} u_0(x) & = & 1+\beta x, \\ u_{k+1}(x) & = & \int_0^x \int_0^t \left((A_k+B_k)u_k(\frac{1}{2}x)\right) dx\, dt, k \geq 0, \end{array} \tag{60}$$

where $\beta = u'(0)$ that will be determined later.

Using (60) we obtain the following approximations

$$\begin{array}{rcl} u_0(x) & = & 1+\beta x, \\ u_1(x) & = & x^2+\beta x^3+\frac{13\beta^2}{24}x^4+\frac{3\beta^3}{20}x^5+\frac{\beta^4}{60}x^6, \\ u_2(x) & = & \frac{1}{24}x^6+\frac{31\beta}{336}x^7+O(x^8), \\ \vdots & & \end{array} \tag{61}$$

The series solution is therefore given by

$$u(x) \;=\; 1+\beta x+x^2+\beta x^3+\beta^2 x^4+\beta^3 x^5+\cdots. \tag{62}$$

To determine β, and hence to obtain the exact solution, we substitute the boundary condition $u(1) = \frac{1}{2}$ into the two-term ϕ_1, three-term ϕ_2, and four-term ϕ_3 approximants, and solving the resulting equations we obtain the following sequence of values for β given by

$$-0.9302270150, -0.9332130352, -0.9862131418, \cdots, \tag{63}$$

and hence this sequence converges to $\beta = -1$. Substituting this value of $\beta = 1$ into (62) leads to the series solutions

$$y(x) = 1-x+x^2-x^3+x^4-x^5+\cdots, \tag{64}$$

which leads to the exact solution

$$u(x) = \frac{1}{x+1}. \tag{65}$$

3.4.2. Using the VIM

A significant feature of the variational iteration method is that it can be used directly without any need to Adomian polynomials. Hence, we use the correction functional for (56) in the form

$$u_{n+1}(x) = u_n(x) + \int_0^x (t-x)\left(u_n^{''}(t) - (u_n^2 + u_n^3)\, u_n(\frac{1}{2}t\right) dt, n \geq 0, \qquad (66)$$

where we used $\lambda = t - x$ as given above.

Considering the given initial values, we can select $y_0(x) = 1 + \beta x$, where $\beta = u'(0)$. Using this selection into (66) we obtain the following successive approximations

$$\begin{aligned} &u_0(x) = 1 + \beta x, \\ &u_1(x) = 1 + \beta x + x^2 + \beta x^3 + \tfrac{13\beta^2}{24}x^4 + \tfrac{3\beta^3}{20}x^5 + \tfrac{\beta^4}{60}x^6, x) \\ &u_2(2) = 1 + \beta x + x^2 + \beta x^3 + \tfrac{13\beta^2}{24}x^4 + \tfrac{3\beta^3}{20}x^5 + \tfrac{\beta^4}{60}x^6, x) \\ &u_3(x) = \beta x - x^2 + (\tfrac{\beta}{6} + \tfrac{1}{3})x^3 - (\tfrac{\beta}{48} + \tfrac{7}{48})x^4 + (\tfrac{\beta}{120} + \tfrac{1}{30})x^5 - (\tfrac{5\beta}{4608} + \tfrac{167}{23040})x^6 \\ &+(\tfrac{23\beta}{107520} + \tfrac{379}{322560})x^7 + O(x^8), \\ &\vdots \end{aligned} \qquad (67)$$

To determine β, and hence to obtain the exact solution, we substitute the boundary condition $u(1) = e^{-1}$ into the approximations u_1, u_2, and u_3, and solving the resulting equations we obtain the following sequence of values for β given by

$$-0.9302270150, -0.9902270153, 0.9999432917, \cdots, \qquad (68)$$

where again this sequences converges to $\beta = -1$. This leads to the exact solution obtained earlier by the Adomian decomposition method.

Second, the variational iteration method can be used in a straightforward manner without any need to the Adomian polynomial required by the Adomian decomposition method.

4. CONCLUSION

In this work, we examined the two-point boundary value problem of pantograph delay differential equation by using the Adomian decomposition method and the variational iteration method. We showed that the two methods are powerful ones. However, the Adomian method works effectively but needs double integral in this type of problems, and requires the use of Adomian polynomials for nonlinear terms. For the variational iteration method, there is a need to determine the Lagrange multiplier, which is characteristic of the problem involved in the work. The variational iteration method can be used in a straightforward manner without any need to the Adomian polynomial required by the Adomian decomposition method. The two methods provide the solution by a convergent series. For concrete problems, where exact solutions are not obtainable, the series solution can be used for numerical needs, and in both methods the numerical approximations give high accuracy approximations by using only few iterations.

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