Romanian Reports in Physics **69**, 103 (2017)

# A FRACTIONAL MODEL OF CONVECTIVE RADIAL FINS WITH TEMPERATURE-DEPENDENT THERMAL CONDUCTIVITY

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*Received October 12, 2016*

*Abstract.* The principal purpose of the present article is to examine a fractional model of convective radial fins having constant and temperature-dependent thermal conductivity. In order to solve fractional order energy balance equation, a numerical algorithm namely homotopy analysis transform method is considered. The fin temperature is derived in terms of thermo-geometric fin parameter. Our method is not limited to the use of a small parameter, such as in the standard perturbation technique. The numerical simulation for temperature and fin tip temperature are presented graphically. The results can be used in thermal design to consider radial fins having both constant and temperature-dependent thermal conductivity.

*Key words*: Thermal conductivity, Radial fins, Fractional energy balance equation, HATM.

## 1. INTRODUCTION

Fins find their uses in different directions of scientific and technological fields such as air-conditioning systems, chemical processing equipments, heat exchangers etc. They are commonly designed for increasing heat transfer between base surface and its environment. There are different techniques to enhance convective heat transfer rate such as increasing heat transfer surface area or heat transfer coefficient. It is well known that the surface area of heat transfer can be increased by attaching the fins made of highly conductive materials on base surface. Furthermore, fin material should have highly thermal conductivity to control the temperature variation from base surface to the tip surfaces of the fin. Linear differential equations are used to describe the convective fins having constant thermal conductivity and temperature distribution of a straight fin [1]. It is well known that the thermal conductivity highly depends on the temperature. The temperature-dependent thermal conductivity can be described in terms of a linear function of the temperature for many scientific and technological processes [2, 3]. Whenever the big temperature gradient exists, then the temperature-dependent thermal conductivity is highly significant and this results in less/more energy transfer. The temperature-dependent thermal conductivity is modeled *via* nonlinear differential equations [4, 5]. The nonlinear fin problem has gained highly attention of scientists and engineers due to its industrial importance and uses in semiconductors, heat exchangers, power generators, electronic components etc. [6]. In this connection, Bartas and Sellers [7] examined the heat-rejecting system made of parallel tubes combined by web plates. Furthermore, Coskun and Atay [8] analyzed the convective straight and radial fins having temperature-dependent thermal conductivity with the aid of variational iteration techniques. Cuce and Cuce [9] examined the temperature and fin efficiency of convective straight fines having temperature-dependent thermal conductivity by application of homotopy perturbation scheme. Chiu and Chen [10] investigated the convective-radiative fins by making use of Adomian's decomposition procedure [11]. Arslanturk [12] made an investigation of optimum design of space radiators having temperature-dependent thermal conductivity with the help of Adomian's decomposition technique. Patra and Ray [13] analyzed the convective radial fins having temperature-dependent thermal conductivity with the aid of homotopy perturbation Sumudu transform approach.

The fractional model of nonlinear equations is useful and it models the temperature-dependent thermal conductivity in a better and systematic manner. It is well known that the future state of a dynamical system depends not only on its present state but also on its past memory. So, the mathematical modeling by the aid of fractional derivatives is more realistic and that is why fractional approaches have gained more and more attention of scientists and engineers. In this connection one can refer the work of Hilfer [14], Mainardi *et al.* [15], Podlubny [16], Caputo [17], Baleanu *et al.* [18], Kilbas *et al.* [19], Kumar *et al.* [20], Singh *et al.* [21], Ma *et al.* [22], Agila *et al.* [23], Abdelkawy *et al.* [24], and Bhrawy *et al.* [25]; see also the recent relevant works [26]-[34]. It is very hard to handle the nonlinear problems of fractional orders. In order to solve such nonlinear problems Liao [35–37] discovered and developed the *homotopy analysis method* (HAM). The HAM has been successfully employed to study various physical phenomena such as fractional model of Black-Scholes equation describing financial markets [38], generalized second-grade fluid fast porous plate [39], Kawahara equation [40], nonlinear Poisson equation [41] etc. The integral transform techniques, specially the Laplace transform, constitute efficient schemes to obtain the solutions of linear differential equations. It is observed that merging of semi-analytical techniques with Laplace transform is very efficient and gives the solution of nonlinear problems in less time. Currently, many mathematicians and engineers have tried to solve the nonlinear equations by employing various schemes merged with the Laplace transform. The merger of Laplace transform and Adomian's decomposition method is used to handle nonlinear differential equations [42], nonlinear integral equations [43] etc. The homotopy perturbation technique is also merged to investigate nonlinear exponential boundary layer equation [44], advection problem [45], fractional Fornberg-Whitham equation [46] etc. In another attempt, the HAM is also merged with Laplace transform algorithm to analyze the nonlinear boundary value problem on semi-infinite domain [47], nonlinear fractional shock wave equation [48], fractional models occurring in unidirectional propagation of long waves through dispersive medium [49] etc.

Nowadays, it is very important and useful to analyze the heat transfer in extended surface and related issues with high accuracy because of growing importance of high performance heat transfer surfaces having lower weights, volumes, initial conditions and handling cost of systems.

In the present paper, we examine the suitability and effectiveness of the *homotopy analysis transform method* (HATM) to find the solution of the fractional model of energy balance equation. Further, we obtain the temperature and fin temperature of convective radial fins having thermal conductivity. The HATM is a merging of HAM, Laplace transform technique, and homotopy polynomials. The proposed technique is free from a small parameter, because it contains an auxiliary parameter $\hbar$ by which we can insure the convergence of series solution and it contains the results obtained by using Adomian decomposition method (ADM), variational interation method (VIM), and homotopy perturbation Sumudu transform method (HPSTM) as particular cases. The plan of the present article is as follows. The basic definitions of calculus of fractional order are discussed in Sec. 2. Section 3 presents the fractional model of convective radial fins having temperature-dependent thermal conductivity. In Sec. 4, we illustrate the applicability of HATM to find the solution of the energy balance equation of fractional order. Section 5 is dedicated to numerical results and discussion. Lastly, the concluding remarks are given in Sec. 6.

## 2. BASIC DEFINITIONS OF FRACTIONAL CALCULUS AND LAPLACE TRANSFORM

Here we present the following basic definitions related to fractional calculus and Laplace transform.

**Definition 1.** The integral of fractional order $\beta > 0$, of a function $\theta(\eta) \in C_\mu, \mu \geq -1$in Riemann-Liouville sense is defined as [16]:

$$J^\beta \theta(\eta) = \frac{1}{\Gamma(\beta)} \int_0^\eta (\eta - \tau)^{\beta-1} \theta(\tau) d\tau, \;\; (\beta > 0), \quad (1)$$

$$J^0 \theta(\eta) = \theta(\eta). \quad (2)$$

**Definition 2.** The fractional order derivative of $\theta(\eta)$ defined by Caputo [17] is given in the following form:

$$D^{\beta}\theta(\eta) = J^{n-\beta}D^{n}\theta(\eta) = \frac{1}{\Gamma(n-\beta)}\int_0^{\eta}(\eta-\tau)^{n-\beta-1}\theta^{(n)}(\tau)d\tau, \tag{3}$$

for $n-1<\beta\le n,\ n\in N,\ \ \eta>0.$

**Definition 3.** If $\theta(\eta)$ is a function, then the Laplace transform formula for $\mathrm{D}^{\beta}\theta(\eta)$ is given by [17, 19]

$$\mathrm{L}\,[\mathrm{D}^{\beta}\theta(\eta)] \,=\, \mathrm{p}^{\beta}\,L[\theta(\eta)] - \sum_{\mathrm{r}=0}^{n-1}\mathrm{p}^{\beta-\mathrm{r}-1}\,\theta^{(\mathrm{r})}(0+),\, (\mathrm{n}\,-1\,<\,\beta\,\le\,\mathrm{n})\,. \tag{4}$$

### 3. CONVECTIVE RADIAL FINS HAVING TEMPERATURE-DEPENDENT THERMAL CONDUCTIVITY

Here we consider the heat pipe/fin space radiator as represented in Fig. 1. Both surfaces of the fin are radiating to the outer space at a very low temperature, which is considered equal to zero absolute. The temperature-dependent thermal conductivity of fin $k$ is a linear function of temperature and the fin is diffuse-grey with emissivity $\varepsilon$. It is assumed that the tube surfaces temperature and the base temperature $T_b$ of the fin are constant, and the radiative exchange between the fin and the heat pipe is negligible. The temperature distribution within the fin is considered to be one dimensional, as the fin is considered to be thin. Therefore, only fin tip length $b$ is assumed as the domain of computation [8].

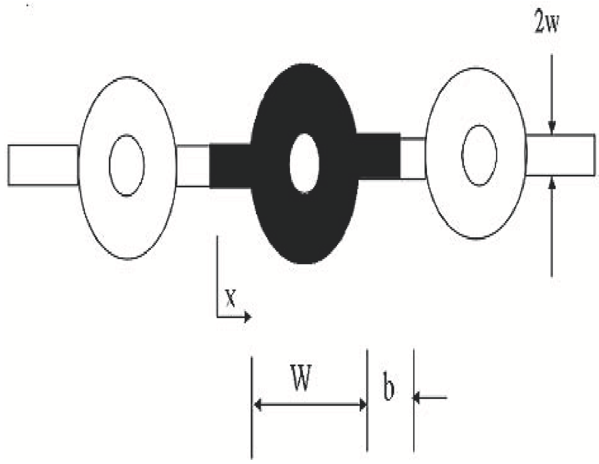


Fig. 1 – Schematic view of the problem.

The mathematical model of a differential element is described via energy balance equation presented in the following form

$$2w\frac{d}{dx}\left[k(T)\frac{dT}{dx}\right]-2\varepsilon\sigma T^4=0. \tag{5}$$

In the above equation $k(T)$ represents the thermal conductivity and $\sigma$ denotes the Stefan-Boltzman constant. It is assumed that the thermal conductivity of the fin material is presented by the following relation

$$k(T)=k_b\left[1+\lambda(T-T_b)\right]. \tag{6}$$

In Eq. (6) $k_b$ indicates the thermal conductivity of the fin at the base temperature and $\lambda$ stands for the slope of the curve drawn between thermal conductivity and temperature.

To solve Eq. (5), we use the subsequent dimensionless variables

$$\theta=\frac{T}{T_b},\eta=\frac{x}{b},\alpha=\lambda T_b,\ \text{and}\ \ \psi=\frac{\varepsilon\sigma b^2T_b^3}{k_bw}. \tag{7}$$

Then Eq. (5) reduces to the following form

$$\frac{d^2\theta}{d\eta^2}+\alpha\theta\frac{d^2\theta}{d\eta^2}+\alpha\left(\frac{d\theta}{d\eta}\right)^2-\psi\theta^4,0\le\eta\le 1 \tag{8}$$

with the boundary conditions

$$\left.\frac{d\theta}{d\eta}\right|_{\eta=0}\ \ \text{and}\ \ \theta|_{\eta=1}=1. \tag{9}$$

Replacing the derivative $d^2\theta/d\eta^2$ in (8) by a fractional derivative of any kind (Riemann-Liouville, Caputo or any other) we may convert (8) to a fractional version of the energy balance equation. In the context of the introductory analysis we assume the case when the fractional order derivative is termed in Caputo sense, namely

$$\frac{d^\beta\theta}{d\eta^\beta}+\alpha\theta\frac{d^2\theta}{d\eta^2}+\alpha\left(\frac{d\theta}{d\eta}\right)^2-\psi\theta^4,0\le\eta\le 1 \tag{10}$$

along with the boundary conditions (9).

The rate of heat transfer from the radial fins is obtained with the help of Newton's law of cooling

$$Q=\int_0^b P(T-T_a)dx. \tag{11}$$

The ratio of the actual heat transfer from the fin surface to ideal heat transfer from the fins, if the fins are completely present in base temperature, is known as the fin efficiency

$$\mu=\frac{Q}{Q_{ideal}}=\frac{\int_0^b P(T-T_a)dx}{Pb(T_b-T_a)dx}=\int_0^1\theta(\eta)d\eta. \tag{12}$$

## 4. HATM FOR NONLINEAR FRACTIONAL ENERGY BALANCE EQUATION

Initially, we apply the Laplace transform on fractional energy balance equation (10) and it gives the result

$$L[\theta(\eta)] - \frac{K}{p} + \frac{1}{p^{\beta}} L\left[\alpha\theta\frac{d^2\theta}{d\eta^2} + \alpha\left(\frac{d\theta}{d\eta}\right)^2 - \psi\theta^4\right] = 0. \tag{13}$$

In view of HAM, we define and represent the nonlinear operator as

$$N[\phi(\eta;q)] = L[\phi(\eta;q)] - \frac{K}{p}$$

$$+\frac{1}{p^{\beta}} L\left[\alpha\phi(\eta;q)\frac{d^2\phi(\eta;q)}{d\eta^2} + \alpha\left(\frac{d\phi(\eta;q)}{d\eta}\right)^2 - \psi\phi^4(\eta;q)\right]. \tag{14}$$

In Eq. (14) $0 \le q \le 1$ is indicating a parameter known as embedding parameter and the expression $\phi(\eta;q)$ is representing a function of variables $\eta$ and $q$. Now with the help of HAM [35–37], the homotopy is constructed as

$$(1-q)\,L\,[\phi(\eta\,;q) - \theta_0(\eta)] = \hbar\, q A(\eta)\, N\,[\phi(\eta\,;q)]. \tag{15}$$

In the above expression $L$ indicates the Laplace transform operator, $A(\eta) \neq 0$ indicates an auxiliary function, $\hbar \neq 0$ stands for an auxiliary parameter, and $\theta_0(\eta)$ denotes an initial guess of $\theta(\eta)$. If we set $q = 0$ and $q = 1$, it gives the results:

$$\phi(\eta;0) = \theta_0(\eta), \qquad \phi(\eta\,;1) = \theta(\eta), \tag{16}$$

respectively. We can see that as the values of $q$ increases from 0 to 1, the solution $\phi(\eta\,;q)$ changes from $\theta_0(\eta)$ to the solution $\theta(\eta)$ of the fractional order energy balance equation. If $\phi(\eta\,;q)$ is expressed in series form about $q$ by the aid of Taylor's theorem, it yields

$$\phi(\eta\,;q) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta)\, q^m, \tag{17}$$

where

$$\theta_m(\eta) = \frac{1}{m!}\frac{\partial^m \phi(\eta\,;q)}{\partial q^m}\,|_{q=0}. \tag{18}$$

If we select the $\hbar$ and $A(\eta)$ in proper way, the series (17) converges at $q = 1$, then we get the result

$$\theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta). \tag{19}$$

The above Eq. (19) must be one of the solutions of the fractional energy balance Eq. (10). Now, we differentiate the Eq. (15) $m$-times with respect to $q$ and further the resulting equation is divided by $m!$, and lastly letting $q = 0$, it gives the subsequent deformation equation of $m$th-order:

$$L\left[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)\right] = \hbar A(\eta) \Re_m(\vec{\theta}_{m-1}). \tag{20}$$

Now using the inversion of Laplace transform operator on Eq. (20), it yields

$$\theta_m(\eta) = \chi_m \theta_{m-1}(\eta) + \hbar L^{-1}[A(\eta)\Re_m(\vec{\theta}_{m-1})]. \tag{21}$$

In Eq. (21), the value of $\chi_m$ is given as

$$\chi_m = 0,\ if\ m \leq 1 \ \text{ and } \ \chi_m = 1,\ if\ m > 1 \tag{22}$$

and $\Re_m(\vec{\theta}_{m-1})$ is presented as

$$\Re_m(\vec{\theta}_{m-1}) = L[\theta_{m-1}(\eta)] - (1-\chi_m)\frac{K}{p}$$

$$+\frac{1}{p^\beta}\left[\alpha B'_{m-1}(\theta_0,\theta_1,...,\theta_{m-1}) + \alpha B''_{m-1}(\theta_0,\theta_1,...,\theta_{m-1}) - \psi B'''_{m-1}(\theta_0,\theta_1,...,\theta_{m-1}\right]. \tag{23}$$

In Eq. (23) $B'_m$, $B''_m$, and $B'''_m$ are the homotopy polynomials [50] given in the following form

$$B'_m = \frac{1}{\Gamma(m)}\left[\frac{\partial^m}{\partial q^m}\left(\phi(\eta;q)\frac{d^2\phi(\eta;q)}{d\eta^2}\right)\right]_{q=0}, \tag{24}$$

$$B''_m = \frac{1}{\Gamma(m)}\left[\frac{\partial^m}{\partial q^m}\left(\frac{d\phi(\eta;q)}{d\eta}\right)^2\right]_{q=0}, \tag{25}$$

$$B'''_m = \frac{1}{\Gamma(m)}\left[\frac{\partial^m}{\partial q^m}\phi^4(\eta;q)\right]_{q=0}, \tag{26}$$

and

$$\phi(\eta,q) = \phi_0 + q\phi_1 + q^2\phi_2 + \cdots. \tag{27}$$

Now using the initial approximation $\theta_0(\eta) = K$ and recursive relation (21), we get the following components of the series solution:

$$\theta_1(\eta) = -\hbar\frac{\psi K^4 \eta^\beta}{\Gamma(\beta+1)}, \tag{28}$$

$$\vdots$$

Similarly, by using of the above process, the components $\theta_m, m \geq 0$ of the HATM solution can be found and consequently the solution can be completely obtained.

Lastly, the HATM solution is presented by the truncated series

$$\theta(\eta) = \lim_{N\to\infty}\sum_{m=0}^{N}\theta_m(\eta). \tag{29}$$

*Table 1*

Comparative study between ADM [4], VIM [8], HPSTM [13], and HATM for the temperature distribution within the fin when $\alpha = 0$, $\psi = 0.2$, and $\beta = 2$.

| $\eta$ | ADM [4] | VIM [8] | HPSTM [13] | HATM |
|---|---|---|---|---|
| 0.0 | 0.923254 | 0.923321 | 0.924241 | 0.9232805443 |
| 0.1 | 0.923981 | 0.924048 | 0.924971 | 0.9240075916 |
| 0.2 | 0.926166 | 0.926234 | 0.927165 | 0.9261933235 |
| 0.3 | 0.929824 | 0.929893 | 0.930832 | 0.9298515954 |
| 0.4 | 0.934978 | 0.935048 | 0.935991 | 0.9350057886 |
| 0.5 | 0.941661 | 0.941732 | 0.942665 | 0.9416892413 |
| 0.6 | 0.949917 | 0.949988 | 0.950885 | 0.9499458541 |
| 0.7 | 0.959803 | 0.959871 | 0.960685 | 0.9598308679 |
| 0.8 | 0.97188 | 0.971446 | 0.972107 | 0.9714118144 |
| 0.9 | 0.984753 | 0.984792 | 0.985196 | 0.9847696396 |
| 1.0 | 1 | 1 | 1 | 1 |

## 5. NUMERICAL RESULTS AND DISCUSSIONS

The present part of this article is dedicated to the numerical simulation for dimensionless temperature distribution $\theta(\eta)$ and dimensionless fin temperature $K$. The comparative study of the obtained results for dimensionless temperature distribution by using different methodologies is presented in Tables 1 and 2.

From Tables 1 and 2, it can be easily observed that the results derived by the application of present technique are in a good agreement with the outcomes existing in the literature. The numerical simulations conducted by employing the HATM for dimensionless temperature distribution $\theta(\eta)$ for different values of $\beta$, $\psi$, and $\alpha$ are depicted in Figs. 2-4, respectively. From Fig. 2, we can observe that as the value of order of time-derivative $\beta$ increases, the value of temperature $\theta(\eta)$ increases. From Fig. 3, it can be noticed that as the value of thermo-geometric fin parameter $\psi$ increases, then the corresponding value of temperature $\theta(\eta)$ decreases. It is worth mentioning that as the value of $\alpha$ increases, then the corresponding value of temperature $\theta(\eta)$ increases; see Fig. 4. The numerical simulations for the dimensionless fin temperature $K$ for different values of $\beta$ and $\alpha$ are represented in Figs. 5 and 6, respectively. From Fig. 5, it can be noticed that as the value of $\beta$ increases, the fin temperature $K$ increases. Figure 6 indicates that as the value of thermal conductivity parameter $\alpha$ increases, then the corresponding value of the fin temperature $K$ increases.

*Table 2*

Comparative study between ADM [4], VIM [8], HPSTM [13], and HATM for the temperature distribution within the fin when $\alpha = 0$, $\psi = 0.5$, and $\beta = 2$.

| $\eta$ | ADM [4] | VIM [8] | HPSTM [13] | HATM |
|---|---|---|---|---|
| 0.0 | 0.852369 | 0.852776 | 0.853142 | 0.8525753653 |
| 0.1 | 0.853690 | 0.854101 | 0.854468 | 0.8538976349 |
| 0.2 | 0.857669 | 0.858087 | 0.858468 | 0.8578809172 |
| 0.3 | 0.864357 | 0.864787 | 0.865204 | 0.8645752409 |
| 0.4 | 0.873839 | 0.874285 | 0.874775 | 0.8740660170 |
| 0.5 | 0.886240 | 0.886704 | 0.887313 | 0.8864770816 |
| 0.6 | 0.901730 | 0.902205 | 0.90297 | 0.9019749578 |
| 0.7 | 0.920529 | 0.920996 | 0.921913 | 0.9207743353 |
| 0.8 | 0.942920 | 0.943336 | 0.944306 | 0.9431447651 |
| 0.9 | 0.969261 | 0.969543 | 0.970296 | 0.9694185748 |
| 1.0 | 1 | 1 | 1 | 1 |

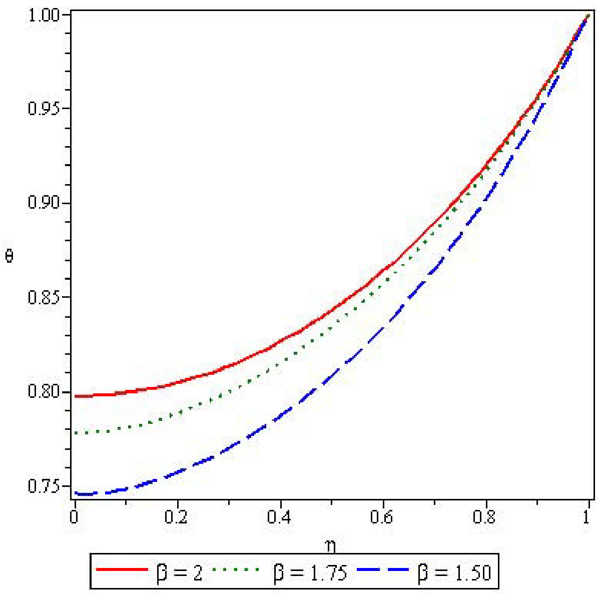


Fig. 2 – Plots of the dimensionless temperature distribution $\theta(\eta)$ of radial fins *vs.* $\eta$ for various values of $\beta$ at $\alpha = 0.5, \psi = 1$, and $h = -1$

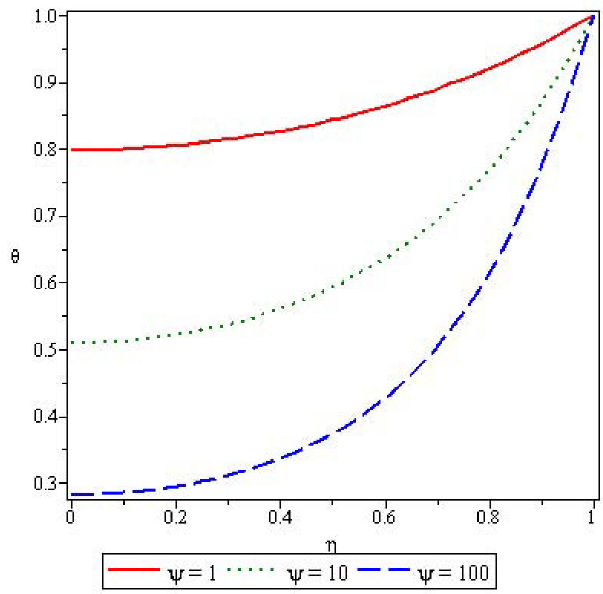


Fig. 3 – Response of the dimensionless temperature distribution $\theta(\eta)$ of radial fins corresponding to $\eta$ for various values of $\psi$ at $\alpha = 0.5$, $\beta = 2$, and $h = -1$.

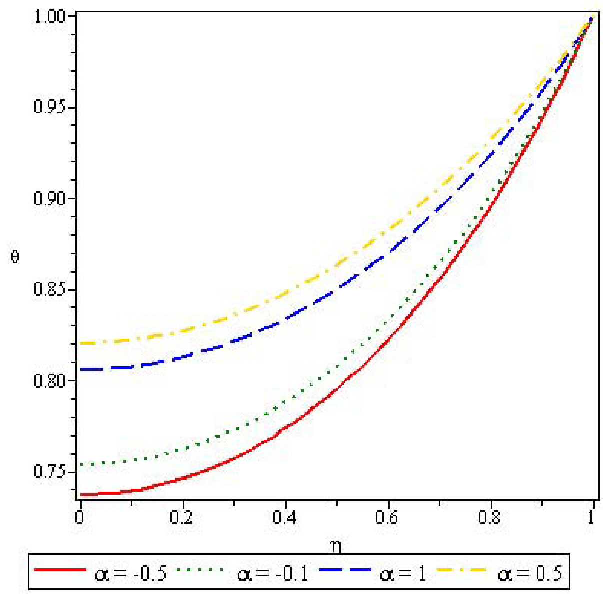


Fig. 4 – Behavior of the dimensionless temperature distribution $\theta(\eta)$ of radial fins *vs.* $\eta$ for different values of $\alpha$ at $\psi = 1$, $\beta = 2$, and $h = -1$.

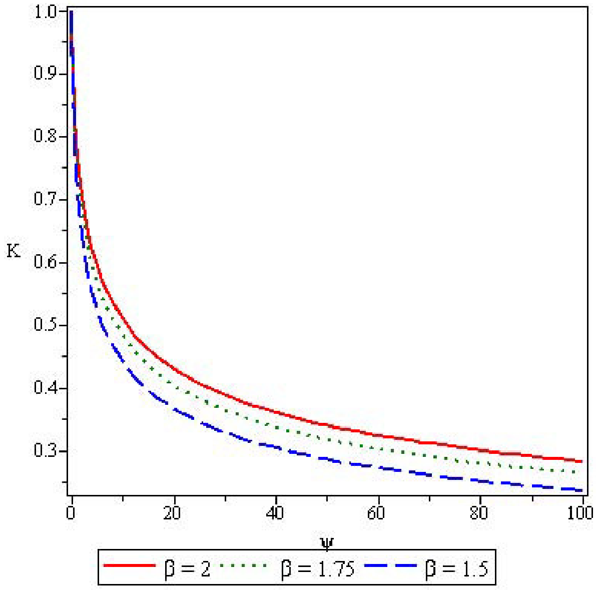


Fig. 5 – Variation of the fin temperature $K$ *vs.* $\psi$ for different values of $\beta$ at $\alpha = 0.2$ and $h = -1$.

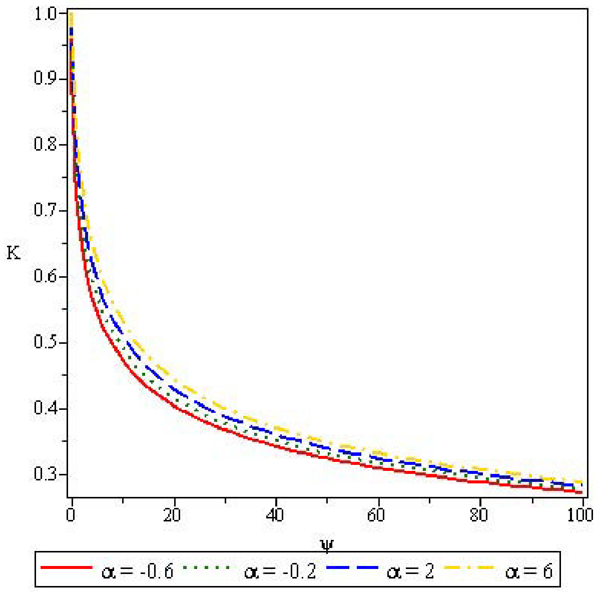


Fig. 6 – Variation of the fin temperature $K$ *vs.* $\psi$ for different values of $\alpha$ at $\beta = 2$ and $h = -1$.

## 6. CONCLUDING REMARKS

In this work, the fractional model of convective radial fins having constant and temperature-dependent thermal conductivity is successfully examined. The fractional energy balance equation is solved by using HATM. The HATM is a very effective computational technique to analyze nonlinear equations of fractional order. The HATM involves an auxiliary parameter $\hbar$ and by using this parameter we can insure the convergence of the solutions. We also analyzed the fractional energy balance equation and related issues. The numerical simulations for the dimensionless temperature distribution $\theta(\eta)$ and the dimensionless fin temperature $K$ are shown graphically and indicate that the order of derivative and the thermo-geometric fin parameter $\psi$ significantly affects the temperature and fin temperature profiles. The outcomes of the present study are very useful for scientists and engineers working with the heat conduction problems having strong nonlinearities. Hence, we can conclude that the suggested technique is very powerful and efficient to investigate the nonlinear fractional equations and fractional models describing the real world problems in a better and systematic manner.

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