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ON THE q-DEFORMED MODIFIED KADOMTSEV-PETVIASHVILI HIERARCHY AND ITS ADDITIONAL SYMMETRIES

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Abstract. The paper aims to construct the q-deformed modified Kadomtsev-Petviashvili (q-mKP) hierarchy and its additional symmetries. The results give the different properties and relationship between the q-mKP hierarchy and the classical mKP hierarchy.

Key words: q-derivative, q-mKP hierarchy, additional symmetries.

1. INTRODUCTION

The q-deformed integrable system (also called q-analogue or q-deformation of the classical integrable system) is performed by using the q-derivative ∂_q [1, 2] to take the place of the ordinary derivative ∂_x in the classical one, where q is a parameter. It is clear that the q-deformed integrable system reduces to the classical integrable system as $q \to 1$ and inherits some integrable structures. Recently, the q-deformed Kadomtsev-Petviashvili (q-KP) hierarchy [3–9] has attracted a lot of interest both in mathematics and physics. As we know, the τ function, sub-hierarchy and some integrable structures of the q-KP hierarchy have already been reported. The extended q-deformed KP hierarchy with self-consistent sources [10, 11] has also been constructed.

Furthermore, the modified KP (mKP) hierarchy [12–14] is an important subject in the field of classical integrable systems. In fact, the KP hierarchy and mKP hierarchy are corresponding to the decompositions of the algebra of pseudo-differential operators

$$\{\sum_i u_i\partial^i\} = \{\sum_{i\geq k} u_i\partial^i\} \oplus \{\sum_{i<k} u_i\partial^i\}$$

for $k = 0, 1$, respectively. In contrast with the well-studied KP hierarchy and mKP hierarchy, the q-deformed mKP hierarchy has not been investigated yet. In this paper, we will give a systematical procedure to construct the q-mKP hierarchy following Sato theory.

It should be noticed that a specific interesting aspect of the research of KP hierarchy is additional symmetry [15–24]. The additional symmetry flows are special symmetries that are not contained in the KP flows and do not commute with each other. The additional symmetry flows of KP hierarchy form the W_∞ algebra [17]. So it is worthy to find the additional symmetries of the q-mKP hierarchy.

The paper is organized as follows. In the end of Sec. 1, we will recall some basic facts on the q-derivative ∂_q. The Lax operator, Lax equation, and flow equations of the q-mKP hierarchy will be presented in Sec. 2. We devote Sec. 3 to deriving the additional symmetries of the q-mKP hierarchy.

In order to get the q-mKP hierarchy, we should introduce some basic facts about the quantum calculus. We denote the q-shift operator and the q-difference operator by θ_q and ∂_q, respectively, where q is a parameter. These operators act on a function $f(x)$ as

$$\begin{aligned}\partial_q f(x) =& \frac{f(qx)-f(x)}{(q-1)x}, \qquad q \neq 1,\\ \theta_q f(x) =& f(qx).\end{aligned}$$

We should note that θ_q does not commute with ∂_q. Indeed,

$$\partial_q \theta_q^n f(x) = q^n \theta_q^n(\partial_q f(x)), \qquad n \in \mathbb{Z}. \tag{1}$$

Let ∂_q^{-1} denote the formal inverse of ∂_q. In general, the Leibnitz rule of ∂_q is

$$\partial_q^n \circ f = \sum_{i=0}^{\infty} C_n^i \theta_q^{n-i}(\partial_q^i f)\partial_q^{n-i}, \qquad n \in \mathbb{Z}, \tag{2}$$

where $\circ$ means composition of operators and the q-binomial, which are defined by

$$\begin{aligned}C_n^0 =& \binom{n}{0}_q = 1,\\ C_n^i =& \binom{n}{i}_q = \frac{(n)_q(n-1)_q\cdots(n-i+1)_q}{(i)_q!},\\ (n)_q =& \frac{1-q^n}{1-q}.\end{aligned}$$

The conjugate operation $*$ for the q-pseudo-differential operator $P = \sum_{-\infty}^{n} p_i \partial_q^i$ is defined by $P^* = \sum\limits_{-\infty}^{n} (\partial_q^*)^i p_i$ with

$$\partial_q^* = -\partial_q \theta_q^{-1} = -\frac{1}{q}\partial_{\frac{1}{q}}, \qquad (\partial_q^{-1})^* = (\partial_q^*)^{-1} = -\theta \partial_q^{-1}.$$

The q-analogue of the classical exponential function $e_q(x)$ is given by

$$e_q(x) = \sum_{i=0}^{\infty} \frac{x^i}{(i)_q!}.$$

Then the formula

$$e_q(x) = \exp(\sum_{j=1}^{\infty} c_j x^j)$$

holds, where

$$c_j = \frac{(1-q)^j}{j(1-q^j)}. \tag{3}$$

As is well known, the function $e_q(x)$ is the eigenfunction of operator ∂_q, that is to say

$$\partial_q e_q(x) = \sum_{j=0}^{\infty} \frac{\partial_q x^j}{(j)_q!} = e_q(x). \tag{4}$$

Considering the function

$$e_q(xz) = \sum_{j=0}^{\infty} \frac{(xz)^j}{(j)_q!},$$

from (4) one obtains immediately

$$\partial_q^k e_q(xz) = z^k e_q(xz), \qquad k \in \mathbb{Z}. \tag{5}$$

The above equation will play a crucial role in q-wave function of q-mKP hierarchy.

2. THE q-DEFORMED MKP HIERARCHY

The main issue that we will investigate is q-mKP hierarchy. Similarly to the general way of describing the classical KP hierarchy, we shall give a brief introduction of q-mKP hierarchy. Let L be one q-ΨDO given by

$$L = u_1\partial_q + u_0 + u_{-1}\partial_q^{-1} + u_{-2}\partial_q^{-2} + \cdots, \tag{6}$$

where $u_i = u_i(x;t_1,t_2,\cdots)$. L is called Lax operator of the q-mKP hierarchy. If we take $q \to 1$ and $u_1 \equiv 1$, the q-mKP hierarchy will reduce to the classical mKP hierarchy. The Lax equation of q-mKP hierarchy is given by

$$\partial_n L = [B_n, L], \qquad \partial_n = \frac{\partial}{\partial t_n}, \tag{7}$$

where t_n are time parameters and $B_n = (L^n)_{\geq 1} = \sum_{i=1}^{n} b_i \partial_q^i$. That is to say B_n is the positive projection of L^n, and $B_n^c = L^n - B_n = (L^n)_{\leq 0}$ is the non-positive projection of L^n. Several explicit forms of B_n can be written out as follows.

$$\begin{aligned}
B_1 =& u_1\partial_q,\\
B_2 =& X_{1,2}\partial_q^2 + X_{1,1}\partial_q,\\
B_3 =& X_{2,3}\partial_q^3 + X_{2,2}\partial_q^2 + X_{2,1}\partial_q,
\end{aligned}$$

where

$$
\begin{aligned}
X_{1,2} =& u_1(\theta_q u_1),\\
X_{1,1} =& u_1(\partial_q u_1) + u_1(\theta_q u_0) + u_0 u_1 + u_{-1}(\theta_q^{-1} u_0) + u_{-2}(\theta_q^{-2} u_1),\\
X_{2,3} =& X_{12}(\theta_q^2 u_1),\\
X_{2,2} =& C_2^1 X_{1,2}(\theta_q(\partial_q u_1)) + X_{1,2}(\theta_q^2 u_0) + X_{1,1}(\theta_q u_1),\\
X_{2,1} =& X_{1,2}(\partial_q^2 u_1) + C_2^1 X_{1,2}(\theta_q(\partial_q u_0)) + X_{1,2}(\theta_q^2 u_{-1})\\
&+ X_{1,1}(\partial_q u_1) + X_{1,1}(\theta_q u_0) + X_{1,0} u_1.
\end{aligned}
$$

The ∂_n-flows are commutative with each other, and we can easily deduce the zero-curvature equation

$$\partial_m B_n - \partial_n B_m + [B_n, B_m] = 0.$$

To illustrate the Lax equation (7) of q-deformed mKP hierarchy, the following examples are given. The first flow equations q-deformed mKP hierarchy are

$$
\begin{aligned}
\partial_1 u_1 =& u_1(\theta_q u_0) - u_0 u_1,\\
\partial_1 u_0 =& u_1(\partial_q u_0) + u_1(\theta_q u_{-1}) - u_{-1}(\theta_q^{-1} u_1),\\
\partial_1 u_{-1} =& u_1(\partial_q u_{-1}) + u_1(\theta_q u_{-2}) - C_{-1}^1 u_{-1}(\theta_q^{-2}(\partial_q u_1)) - u_{-2}(\theta_q^{-2} u_1).
\end{aligned}
$$

The second flow equations are

$$
\begin{aligned}
\partial_2 u_1 =& X_{1,2}(\partial_q^2 u_1) + C_2^1 X_{1,2}(\theta_q(\partial_q u_0)) + X_{1,2}(\theta_q^2 u_{-1})\\
&+ X_{1,1}(\partial_q u_1) + X_{1,1}(\theta_q u_0) - u_1(\partial_q X_{1,1}) - u_0 X_{1,1} - u_{-1}(\theta_q^{-1} X_{1,2}),\\
\partial_2 u_0 =& X_{1,2}(\partial_q^2 u_0) + C_2^1 X_{1,2}(\theta_q(\partial_q u_{-1})) + X_{1,2}(\theta_q^2 u_{-2}) + X_{1,1}(\partial_q u_0)\\
&+ X_{1,1}(\theta_q u_{-1}) - C_{-1}^1 u_{-1}(\theta_q^{-2}(\partial_q X_{1,2})) - u_{-1}(\theta_q^{-1} X_{1,1})\\
&- u_{-2}(\theta_q^{-2} X_{1,2}),\\
\partial_2 u_{-1} =& X_{1,2}(\partial_q^2 u_{-1}) + C_2^1 X_{1,2}(\theta_q(\partial_q u_{-2})) + X_{1,2}(\theta_q^2 u_{-3}) + X_{1,1}(\partial_q u_{-1})\\
&+ X_{1,1}(\theta_q u_{-2}) - C_{-1}^2 u_{-1}(\theta_q^{-3}(\partial_q^2 X_{1,2})) - C_{-1}^1 u_{-1}(\theta_q^{-2}(\partial_q X_{1,1}))\\
&- C_{-2}^1 u_{-2}(\theta_q^{-3}(\partial_q X_{1,2}) - u_{-2}(\theta_q^{-2} X_{1,1}) - u_{-3}(\theta_q^{-3} X_{1,2}).
\end{aligned}
$$

The third flow equations are

$$\begin{aligned}
\partial_3 u_1 =& X_{2,3}(\partial_q^3 u_1) + C_3^2 X_{2,3}(\theta_q(\partial_q^2 u_0)) + C_3^1 X_{2,3}(\theta_q^2(\partial_q u_{-1})) + X_{2,3}(\theta_q^3 u_{-2}) \\
&+ X_{2,2}(\partial_q^2 u_1) + C_2^1 X_{2,2}(\theta_q(\partial_q u_0)) + X_{2,2}(\theta_q^2 u_{-1}) + X_{2,1}(\partial_q u_1) \\
&+ X_{2,1}(\theta_q u_0) - u_1(\partial_q X_{2,1}) - u_0 X_{2,1} - C_{-1}^1 u_{-1}(\theta_q^{-2}(\partial_q X_{2,3})) \\
&- u_{-1}(\theta_q^{-1} X_{2,2}) - u_{-2}(\theta_q^{-2} X_{2,3}), \\
\partial_3 u_0 =& X_{2,3}(\partial_q^3 u_0) + C_3^2 X_{2,3}(\theta_q(\partial_q^2 u_{-1})) + C_3^1 X_{2,3}(\theta_q^2(\partial_q u_{-2})) + X_{2,3}(\theta_q^3 u_{-3}) \\
&+ X_{2,2}(\partial_q^2 u_0) + C_2^1 X_{2,2}(\theta_q(\partial_q u_{-1})) + X_{2,2}(\theta_q^2 u_{-2}) + X_{2,1}(\partial_q u_0) \\
&+ X_{2,1}(\theta_q u_{-1}) - C_{-1}^2 u_{-1}(\theta_q^{-3}(\partial_q^2 X_{2,3})) - C_{-1}^1 u_{-1}(\theta_q^{-2}(\partial_q X_{2,2})) \\
&- u_{-1}(\theta_q^{-1} X_{2,1}) - C_{-2}^1 u_{-2}(\theta_q^{-3}(\partial_q X_{2,3})) - u_{-2}(\theta_q^{-2} X_{2,2}) \\
&- u_{-3}(\theta_q^{-3} X_{2,3}), \\
\partial_3 u_{-1} =& X_{2,3}(\partial_q^3 u_{-1}) + C_3^2 X_{2,3}(\theta_q(\partial_q^2 u_{-2})) + C_3^1 X_{2,3}(\theta_q^2(\partial_q u_{-3})) \\
&+ X_{2,2}(\partial_q^2 u_{-1}) + C_2^1 X_{2,2}(\theta_q(\partial_q u_{-2})) + X_{2,2}(\theta_q^2 u_{-3}) + X_{2,1}(\partial_q u_{-1}) \\
&+ X_{2,1}(\theta_q u_{-2}) - C_{-1}^3 u_{-1}(\theta_q^{-4}(\partial_q^3 X_{2,3})) - C_{-1}^2 u_{-1}(\theta_q^{-3}(\partial_q^2 X_{2,2})) \\
&- C_{-1}^1 u_{-1}(\theta_q^{-2}(\partial_q X_{2,1})) - C_{-2}^2 u_{-2}(\theta_q^{-4}(\partial_q^2 X_{2,3})) + X_{2,3}(\theta_q^3 u_{-4}) \\
&- C_{-2}^1 u_{-2}(\theta_q^{-3}(\partial_q X_{2,2})) - u_{-2}(\theta_q^{-2} X_{2,1}) - C_{-3}^1 u_{-3}(\theta_q^{-4}(\partial_q X_{2,3})) \\
&- u_{-3}(\theta_q^{-3} X_{2,2}) - u_{-4}(\theta_q^{-4} X_{2,3}).
\end{aligned}$$

Obviously, $\partial_{t_1} = \partial_x$ and flow equations are reduced to classical mKP flows when $q \to 1$.

The Lax operator in (6) can be represented in a dressing form

$$L = \phi \partial_q \phi^{-1}, \tag{8}$$

where ϕ is a q-ΨDO and

$$\phi = \sum_{i=0}^{\infty} w_{-i} \partial_q^{-i}. \tag{9}$$

We should note that $w_0 \neq 1$. It is obviously that the dressing operator of the q-mKP hierarchy is different from both the classical mKP hierarchy and the classical KP hierarchy based on ∂_q and $w_0 \neq 1$. The dressing operator ϕ is determined up to the multiplication on the right by a series in ∂_q^{-1} with constant coefficients $\sum\limits_{i=0}^{\infty} c_{-i} \partial_q^{-i}$. Equation (9) implies expressing all u_i in terms of differential polynomials in w_i. The expressions of u_i in terms of w_i have the forms

$$\begin{aligned}
w_0 =& u_1(\theta_q w_0),\\
w_{-1} =& u_1(\theta_q w_{-1}) + u_1(\partial_q w_0) + u_0 w_0,\\
w_{-2} =& u_1(\theta_q w_{-2}) + u_1(\partial_q w_{-1}) + u_0 w_{-1} + u_{-1}(\theta_q^{-1} w_0),\\
w_{-3} =& u_1(\theta_q w_{-3}) + u_1(\partial_q w_{-2}) + u_0 w_{-2} + C_{-1}^1 u_{-1}(\theta_q^{-2}(\partial_q w_0))\\
&+ u_{-2}(\theta_q^{-2} w_0) + u_{-1}(\theta^{-1} w_{-1}),\\
w_{-4} =& u_1(\theta_q w_{-4}) + u_1(\partial_q w_{-3}) + u_0 w_{-3} + C_{-1}^2 u_{-1}(\theta_q^{-3}(\partial_q^2 w_0))\\
&+ C_{-1}^1 u_{-1}(\theta_q^{-2}(\partial_q w_{-1})) + u_{-1}(\theta_q^{-1} w_{-2}) + u_{-2}(\theta_q^{-2} w_{-1})\\
&+ C_{-2}^1 u_{-2}(\theta_q^{-3}(\partial_q w_0)) + u_{-3}(\theta_q^{-3} w_0).
\end{aligned}$$

Theorem 2.1 *The dressing operator ϕ satisfies the Sato equation*

$$\partial_n \phi = -B_n^c \phi, \qquad n \in \mathbb{Z}^+. \tag{10}$$

Proof. From the Lax equation (7), we can get

$$\begin{aligned}
\partial_n L &= [B_n, L] = B_n L - L B_n\\
&= (L^n - B_n^c)L - L(L^n - B_n^c)\\
&= -[B_n^c, L].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\partial_n L &= \partial_n(\phi \partial_q \phi^{-1})\\
&= (\partial_n \phi)\partial_q \phi^{-1} + \phi \partial_q (\partial_n \phi^{-1})\\
&= (\partial_n \phi)\phi^{-1}\phi \partial_q \phi^{-1} - \phi \partial_q \phi^{-1}(\partial_n \phi)\phi^{-1}\\
&= (\partial_n \phi)\phi^{-1} L - L(\partial_n \phi)\phi^{-1}\\
&= [(\partial_n \phi)\phi^{-1}, L].
\end{aligned}$$

Hence

$$-[B_n^c, L] = [(\partial_n \phi)\phi^{-1}, L].$$

The above equation implies that

$$\partial_n \phi = -B_n^c \phi, \qquad n \in \mathbb{Z}^+,$$

which ends the proof.

Let $\xi(t,z) = \sum_{i=1}^{\infty} t_i z^i$, the operator ∂_m acting on the function $\xi(t,z)$ reads

$$\begin{aligned}
\partial_m \xi(t,z) =& z^m,\\
\partial_m \exp[\xi(t,z)] =& z^m \exp[\xi(t,z)], \qquad m \in \mathbb{Z}^+.
\end{aligned}$$

Definition 2.2 *The q-wave (or q-Baker) function and q-adjoint wave function for q-mKP hierarchy are defined by*

$$w_q(x,t,z) = \phi e_q(xz)\exp[\xi(t,z)]$$

and

$$w_q^*(x,t,z) = (\phi^*)^{-1}|_{1/q} e_{1/q}(-xz)\exp[-\xi(t,z)], \tag{11}$$

where $t = (t_1, t_2, t_3, \cdots)$. *For the q-pseudo-differential* $P = \sum_{-\infty}^{n} p_i \partial_q^i$, *the notation* $P|_{1/q} = \sum_{-\infty}^{n} p_i(\frac{x}{q}) q^i \partial_q^i$ *is used.*

Theorem 2.3 *The q-Baker function satisfies the equations*

$$Lw_q = zw_q, \qquad \partial_m w_q = B_m w_q.$$

Proof. Indeed, using the equation (5), one can obtain

$$\begin{aligned} Lw_q &= \phi\partial_q\phi^{-1}\phi e_q(xz)\exp[\xi(t,z)] \\ &= \phi\partial_q e_q(xz)\exp[\xi(t,z)] \\ &= zw_q. \end{aligned}$$

With the Sato equation (10), it follows that

$$\begin{aligned} \partial_m w_q &= (\partial_m\phi)e_q(xz)\exp[\xi(t,z)] + \phi e_q(xz)(\partial_m \exp[\xi(t,z)] \\ &= -B_m^c \phi e_q(xz)\exp[\xi(t,z)] + z^m \phi e_q(xz)\exp[\xi(t,z)] \\ &= (-B_m^c + L^m)w_q \\ &= B_m w_q. \end{aligned}$$

Theorem 2.4 *The q-adjoint wave function satisfies the equations*

$$(L^*|_{x/q} w_q^*) = zw_q^*, \qquad \partial_m w_q^* = -(B_m|_{x/q})^* w_q^*.$$

Proof. The proof is similar to that of Theorem 2.3.

3. ADDITIONAL SYMMETRIES OF THE q-mKP HIERARCHY

The central task of this Section is to find the additional symmetry flows of the q-mKP hierarchy. We have seen that

$$(\partial_k - B_k) = \phi(\partial_k - \partial_q^k)\phi^{-1},$$

where $B_k = (L^k)_{\geq 1}$. First, let us define Orlov-Shulman's M_q operator and Γ_q.

Definition 3.1 *The operators Γ_q and M_q of q-mKP hierarchy are defined by*

$$\Gamma_q = \sum_{i=1}^{\infty}(it_i + ic_i x^i)\partial_q^{i-1} \tag{12}$$

and

$$M_q = \phi\Gamma_q\phi^{-1}, \tag{13}$$

where c_i is given in Eq. (3)*.*

Dressing the relation $[\partial_k - \partial_q^k, \Gamma_q] = 0$ we obtain the equation $[\partial_k - B_k, M_q] = 0$, or

$$\partial_k M_q = [B_k, M_q]. \tag{14}$$

Definition 3.2 *The additional flows of q-mKP hierarchy are defined by their actions on the dressing operator ϕ*

$$\partial_{lm}^* \phi = -A_{lm}^c \phi, \tag{15}$$

where $\partial_{lm}^*\phi = \frac{\partial\phi}{\partial t_{lm}^*}$ symbolizes a derivative with respect to some additional variable t_{lm}^* and $A_{lm}^c = (M_q^m L^l)_{\leq 0}$.

Theorem 3.3 *The additional flows act on operators Γ_q and M_q of the q-mKP hierarchy as*

$$\partial_{lm}^* L = -[A_{lm}^c, L],$$
$$\partial_{lm}^* M_q = -[A_{lm}^c, M_q].$$

Proof. Using the equations (8) and (15), we have

$$\begin{aligned}
\partial_{lm}^* L &= (\partial_{lm}^*\phi)\partial_q\phi + \phi\partial_q(\partial_{lm}^*\phi^{-1}) \\
&= (\partial_{lm}^*\phi)\partial_q\phi^{-1} - \phi\partial_q(\phi^{-1}(\partial_{lm}^*\phi)\phi^{-1}) \\
&= -A_{lm}^c L + L A_{lm}^c \\
&= -[A_{lm}^c, L].
\end{aligned}$$

Similarly, with the equations (12) and (15) we can get

$$\begin{aligned}
\partial_{lm}^* M_q &= (\partial_{lm}^*\phi)\Gamma_q\phi + \phi\Gamma_q(\partial_{lm}^*\phi^{-1}) \\
&= (\partial_{lm}^*\phi)\Gamma_q\phi^{-1} - \phi\Gamma_q(\phi^{-1}(\partial_{lm}^*\phi)\phi^{-1}) \\
&= -A_{lm}^c M_q + M_q A_{lm}^c \\
&= -[A_{lm}^c, M_q].
\end{aligned}$$

In the above calculation, Γ_q does not depend on the additional flows variables t_{lm}^*.

Corollary 3.4

$$\begin{aligned}
\partial_{lm}^* L^k &= -[A_{lm}^c, L^k], \\
\partial_{lm}^* M_q^k &= -[A_{lm}^c, M_q^k], \\
\partial_{lm}^* M_q^n L^k &= -[A_{lm}^c, M_q^n L^k], \\
\partial_m M_q^n L^k &= [B_m, M_q^n L^k].
\end{aligned}$$

Proof. This is a simple corollary of the Definition 3.2 and Theorem 3.3. We present only the proof of the third equation.

$$\begin{aligned}
\partial_{lm}^* M_q^n L^k =&(\partial_{lm}^* M_q) M_q^{n-1} L^k + \cdots + M_q^{n-1} (\partial_{lm}^* M_q) L^k \\
&+ M_q^n (\partial_{lm}^* L) L^{k-1} + \cdots + M_q^n L^{k-1} (\partial_{lm}^* L) \\
=& - [A_{lm}^c, M_q] M_q^{n-1} L^k + \cdots - M_q^{n-1} [A_{lm}^c, M_q] L^k \\
&- M_q^n [A_{lm}^c, L] L^{k-1} + \cdots - M_q^n L^{k-1} [A_{lm}^c, L] \\
=& - [A_{lm}^c, M_q^n L^k].
\end{aligned}$$

Theorem 3.5 *The operators ∂_{lm}^* commute with all ∂_k , i.e. they indeed determine symmetries as*

$$[\partial_{lm}^*, \partial_k] = 0. \tag{16}$$

The operators ∂_{lm}^* do not commute with each other. We call them additional symmetries of q-mKP hierarchy.

Proof. We have

$$\begin{aligned}
[\partial_{lm}^*, \partial_k]\phi &= -\partial_{lm}^* B_k^c \phi + \partial_k A_{lm}^c \phi \\
&= [A_{lm}^c, L^k]_{\leq 0}\phi + B_k^c A_{lm}^c \phi + [B_k, M_q^m L^l]_{\leq 0}\phi - A_{lm}^c B_k^c \phi \\
&= [A_{lm}^c, L^k]_{\leq 0}\phi - [A_{lm}^c, B_k^c]\phi + [B_k, M_q^m L^l]_{\leq 0}\phi \\
&= [A_{lm}^c, B_k]_{\leq 0}\phi + [B_k, M_q^m L^l]_{\leq 0}\phi \\
&= 0.
\end{aligned}$$

Note that $[\partial_{lm}^*, \partial_{kn}^*]\phi \neq 0$. In fact,

$$\begin{aligned}
[\partial_{lm}^*, \partial_{kn}^*]\phi &= -\partial_{lm}^* A_{kn}^c \phi + \partial_{kn}^* A_{lm}^c \phi \\
&= [A_{lm}^c, M_q^n L^k]_{\leq 0}\phi + A_{kn}^c A_{lm}^c \phi - [A_{kn}^c, M_q^m L^l]_{\leq 0}\phi - A_{lm}^c A_{kn}^c \phi \\
&= [A_{lm}^c, M_q^n L^k]_{\leq 0}\phi - [A_{lm}^c, A_{kn}^c]\phi - [A_{kn}^c, M_q^m L^l]_{\leq 0}\phi \\
&= [A_{lm}^c, A_{kn}]_{\leq 0}\phi - [A_{kn}^c, M_q^m L^l]_{\leq 0}\phi \\
&= [M_q^m L^l, M_q^n L^k]_{\leq 0}\phi \\
&\neq 0.
\end{aligned}$$

The transformations

$$[A^c_{lm}, B^c_k] = [A^c_{lm}, B^c_k]_{\leq 0}$$

and

$$[A^c_{lm}, A_{kn}]_{\leq 0} = [M^m_q L^l, A_{kn}]_{\leq 0}$$

have been used in the above derivation.

4. CONCLUSIONS AND DISCUSSION

To summarize, the q-deformed mKP hierarchy and its additional symmetry flows have been constructed by a modification of the corresponding classical mKP hierarchy. In this process, the q-derivative ∂_q plays a crucial role. In this paper, we have derived the property of q-derivative ∂_q in Sec. 1. The q-mKP hierarchy was discussed in Sec. 2. The Sato equation has been given in Eq. (10). Based on the analytic property of the q-exponent $e_q(x)$, the wave function of q-mKP hierarchy was constructed. The additional symmetries of q-mKP hierarchy in Eq. (15) were obtained in Sec. 3. Our results show that q-mKP hierarchy has, indeed, different properties related to Lax equation and additional symmetries comparing with the classical mKP hierarchy. Finally, we observe that q-mKP hierarchy tends to the classical mKP hierarchy as $q \to 1$.

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