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NEW SOLUTIONS OF DODD-BULLOUGH-MIKHAILOV EQUATION BY USING AN IMPROVED TANH-METHOD

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Abstract. In this paper an improved tanh-method and the symbolic Maple computations are applied to the Dodd–Bullough–Mikhailov (DBM) equation in order to construct in an unitary way several multiple travelling wave solutions. The main idea of this method is to take a full advantage of Riccati equation and consequently to provide a guideline able to classify the various types of solutions accordingly to the values of some parameters. Multiple soliton-like solutions as well as others under trigonometric function forms or under rational forms are discussed. This method may as well successfully solve other nonlinear partial differential equations.

Key words: Generalized-tanh method, symbolic computation, Dodd-Bullough-Mikhailov equation, multiple solutions.

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1. INTRODUCTION

Many matters of physics occurring in science and engineering may be modelled through nonlinear partial differential equations (NPDEs). In order to better understand the concerned phenomena, to search for efficient algorithms able to discover a rich variety of NPDEs solutions, this path has attracted a considerable attention, partly due to the availability of computer symbolic systems like Maple and Mathematica, which do allow to perform some rather complicated and tedious algebraic calculations. During the recent years, various analytical methods have been developed by many scientists in order to find exact and explicit solutions for NPDEs. Among them: the inverse scattering method [1], the Bäcklund transformation [2], the Darboux transformation [3], Hirota's bilinear method [4], the dressing method [5], the homogeneous balance method [6], the (G'/G)-expansion method [7], the Lie symmetry reduction [8–12], the generalized conditional symmetry method [13–15], the sine-cosine method [16], various extended tanh-methods [17–19] and others.

In this paper an improved tanh-function method is applied in order to obtain several multiple solutions for the Dodd-Bullough-Mikhailov (DBM) equation in an unified way. The key idea of this method is to take a full advantage of Riccati equation involving three parameters and to make use of its solutions in order to apply an improved version of tanh-function method. It is quite interesting that relationships between parameters could be used in order to exactly evaluate the numbers and types of the explicit wave solutions. In fact, we will apply this formalism to some equations belonging to a larger class of equation of the form:

$$u_{xt} + pe^{u} + qe^{-2u} = 0, \tag{1}$$

It is a special nonlinear equation which does appear in many problems standing from fluids' flow to quantum field theory. Due to its non-polynomial form, it could not be directly dealt through the tanh-method or the extended tanh-method. In [20] the standard tanh-method combined with Painlevé transformation are employed in order to handle the DBM equation and Liouville equation. The exp-function method [21] is also proposed in order to seek solitary solutions, periodic solutions and compacton-like solutions for the concerned model. Some types of exact travelling wave solutions are derived in [22] by making use of the $\left(\frac{G'}{G}\right)$-expansion method. By making use of the integral bifurcation method, a generalized Tzitzéica-Dodd-Bullough-Mikhailov (TDBM) equation is studied by [23]. With various parameters, different kinds of exact travelling wave solutions are investigated. Many singular travelling wave solutions under blow-up forms or broken forms, such as periodic blow-up wave solutions, solitary wave solutions of blow-up form, broken solitary wave solutions, broken kink wave solutions, and some unboundary wave solutions, are obtained.

In this paper an efficient generalized tanh-method able to simultaneously construct several multiple travelling wave solutions of NPDEs is described in Section 2. By making use of a 'pre-possessing' technique, the algorithmic method with a symbolic computation is applied upon the DBM equation in Section 3. A rich variety of solutions which includes solitons, periodic solutions or rational wave solutions are obtained both for (1) as well as for Liouville equation. Finally, some essential facts are pointed out in concluding remarks.

2. METHOD SUMMARY

In this section we do describe a generalized tanh-method able to construct a rich variety of soliton solutions and periodic solutions for NPDEs. It does extend the general ansatz [24] and it does simply proceed as follows:

Step 1: Consider a nonlinear PDE for the physical field $u(t,x,u)$ given by:

$$E(u,\ u_x, u_t,\ u_{2x},\ u_{xt},\ u_{tt}, ...) = 0. \tag{2}$$

We assume the fact that the solutions of Eq. (2) could be expressed under the form:

$$u(t,x) = \sum_{i=0}^{N} f_i(t,x)\varphi^i(\xi),\ \xi = \xi(t,x), \tag{3}$$

where the functions $\xi(t,x)$, $f_i(t,x)$, $i = \overline{0,N}$ and the positive integer N are to be determined later. The crucial idea is the one of inserting the variable $\varphi(\xi)$ as a solution of Riccati equation:

$$\varphi^{'} = A + B\varphi + C\varphi^2, \tag{4}$$

where "prime" denotes $d/d\xi$. The advantage is the one that the signs of parameters A, B, C may be used in order to exactly evaluate the respective amounts and types of the travellling wave solutions.

Step 2: Determine N in Eq. (3) by balancing the linear term of highest order with the highest nonlinear term in Eq. (2).

Step 3: Substituting (3) and (4) into (2) yields a set of PDEs for $f_i(t,x)$, $i = \overline{0,N}$ and $\xi(t,x)$, because all of the coefficients of φ^i have to vanish.

Step 4: Solving the previous system of PDEs. When substituting its solutions into Eq. (3) and making use of the special solutions for the desired Riccati equation (4), gives a series of soliton solutions, periodic solutions and rational solutions for Eq. (2).

We do remember the fact that Riccati equation (4) admits several types of solutions as, follows:

Case I: If $B = 0$ and $A = -C$, $A \in \{1/2,\ 1\}$, we respectively have the soliton solutions:

$$\varphi(\xi) \in \{\coth\xi \pm \xi,\ \tanh\xi \pm i\,\mathrm{sech}\,\xi\} \text{ or}$$
$$\varphi(\xi) \in \{\tanh\xi/(1 \pm \mathrm{sech}\,\xi),\ \coth\xi/(1 \pm i\xi)\},\ \text{when } A = 1/2$$

and

$$\varphi(\xi) \in \{\tanh\xi,\ \coth\xi\},\ \text{when } A = 1. \tag{5}$$

Case II: If $B = 0$ and $A = C \in \{-1/2,\ 1/2, -1, 1\}$, we respectively outline the periodic solutions:

$$\begin{gathered} \varphi(\xi) \in \{\cot\xi \pm \csc\xi,\ \sec\xi - \tan\xi,\ \cot\xi/(1 \pm \csc\xi)\}, \\ \varphi(\xi) \in \{\tan\xi \pm \sec\xi,\ \csc\xi - \cot\xi,\ \tan\xi/(1 \pm \sec\xi)\}, \\ \varphi(\xi) = \cot\xi,\ \varphi(\xi) = \tan\xi. \end{gathered} \tag{6}$$

Case III: If $A = \pm 1$, $B = \pm C \neq 0$, we get the solutions:

$$\varphi(\xi) = \tan\xi/(1 \mp \tan\xi),\ \text{when } A = 1,\ C = 2, \tag{7}$$

$$\varphi(\xi) = \cot\xi/(1 \mp \cot\xi) \text{ when } A = -1,\ C = -2. \tag{8}$$

Case IV: If $B \neq 0$ and $C = 0$, an exponential-type solution results:

$$\varphi(\xi) = (e^{B\xi} - A)/B. \tag{9}$$

Case V: If $A = B = 0$ and $C \neq 0$, a rational-type solution appears:

$$\varphi(\xi) = -1/(C\xi + c_0),\ c_0 = const. \tag{10}$$

3. APPLICATION UPON THE DBM EQUATION

3.1. INTEGRABLE MODELS OF TODA TYPE

The equation (1) we are going to consider in this paper belongs to a larger class of equations of the form:

$$u_{\tau\tau} - u_{\phi\phi} + f(u) = 0. \tag{11}$$

Alternatively, by considering new variables $t = \tau - \phi,\ x = \tau + \phi$, the equation (11) takes the form:

$$u_{xt} + f(u) = 0. \tag{12}$$

It was established that many physical phenomena coming from solid-state physics, nonlinear optics or quantum field theory are described by equations of this form. The choice of $f(u)$ in an exponential form leads to many interesting and famous equations, all belonging to the integrable Toda field theories. It was considered for the first time, more than one century ago, by the Romanian mathematician G. Tzitzeica [25] in a context related to special surfaces in differential geometry for which the ratio K/d^4 is constant, where K is the Gaussian curvature and d is the distance from the origin to the tangent plane at an arbitrary point of the surface. Let us consider the most general exponential form:

$$u_{xt} + pe^{mu} + qe^{nu} = 0, \tag{13}$$

with p, q, m, n arbitrary constants. The equation (13) with this choice is known as the generalized Tzitzeica-Dodd-Bullough-Mikhailov equation [23]. It is clear now that (1) corresponds to $p = q = m = 1,\ n = -2$. Other highly considered cases are: Liouville equation ($p = -1,\ m = 1,\ q = 0$), Tzitzeica equation [25] ($p = -q = -1,\ m = 1,\ n = -2$), Bullough-Dodd equation ($p = -q = 1,\ m = 1,\ n = -2$) sine-Gordon equation ($p = -q = -1/2,\ m = -n = i$), sinh-Gordon ($p = -q = -1/2,\ m = -n = 1$), Tzitzeica-Dodd-Bullough equation ($p = q = -1,\ m = 1,\ n = -2$). All these equation were studied from many perspectives: from the symmetry point of view, this general equation was analized by Sophus Lie in its famous book [26], and more recently in [27]. In [28] has been proved that it admits an infinity number of polynomial conserved densities if, and only if $n = -m$, for arbitrary complex values for

p and q. Zhiber and Shabat [29] proved that the Tzitzeica case corresponds to a completely integrable Hamiltonian system. Mikhailov extended the Bullogh-Dodd case to the most general DBM equation and constructed its Lax pair [30, 31] with an highly symmetry called the group of reductions. Interesting contributions related to the spectral properties of the Lax operators related to Tzitzeica case were analyzed in [32]. The integrability of a deformed version of the exactly integrable Bullough-Dodd equation was recently considered in [33].

We have to mention that equation (12) can be seen as the Euler-Lagrange equation generated by the Lagrangian:

$$L = -\frac{u_x u_t}{2} + V(u);\ f(u) = \frac{dV}{du}. \tag{14}$$

Coming back to Eq. (1), we have to point out that numerous investigations lead to various type of solutions. The most frequent form of these solutions are expressed in terms of hyperbolic, elliptic or periodic functions. Soliton-type solutions, proportional with $\tanh^2$ and sech^2 appear in [34], and proportional with $\cosh^{-1}$ in [23]. Periodic solutions in terms of armonic functions $\tan^2$ and $\sec^2$ are mentioned in [20] and in [34]. Solutions expressed in terms of Jacobi elliptic functions are figured out in [23]. We will see that our approach will bring a more rich variaty of hyperbolic and periodic solutions.

3.2. THE TANH METHOD APPLIED TO DBM

In this subsection, we do illustrate the efficiency of the algorithmic method mentioned in section 2 upon the DBM class of equations. The algorithm does include and extend the results reported in [24]. Two extensions are considered: (i) the class of solutions for Riccati equation is enlarged by considering its dependence on three parameters instead of one; (ii) the solutions of DBM equation are expressed in terms of Riccati solutions through finite series with variable coefficients, instead of constant ones as they are considered in [24]. Due to the transformation $u = \ln v$, the master Eq. (1) takes the form:

$$v_{xt} v - v_x v_t + p v^3 + q = 0. \tag{15}$$

Balancing the term $v_{xt} v$ with the term v^3 in (15), we obtain $N = 2$. Therefore, we are able to choose the following ansatz:

$$v(t,x) = f(t,x) + g(t,x)\varphi(\xi) + h(t,x)\varphi^2(\xi), \tag{16}$$

where $\xi = mx + \rho(t)$, $m = const$. By substituting (16) into (15) along with Eq. (4), and by setting in the obtained expression the coefficients of $\varphi^k(\xi)$, $k = \overline{0,6}$ to zero, we could deduce, the following set of PDEs with the respective unknown functions $f(t,x)$, $g(t,x)$, $h(t,x)$, $\rho(t)$, namely:

$$h^2(2mC^2\dot{\rho}+ph)=0,$$
$$h\left[g(2mC^2\dot{\rho}+ph)+2mC(Cg+Bh)\dot{\rho}+2pgh\right]=0,$$

$$\begin{aligned}3fh(2mC^2\dot{\rho}+ph)+g^2(mC^2\dot{\rho}+3ph)+8mCh^2(A-B)\dot{\rho}\\ -mChg(4C-13B)\dot{\rho}+(g-2h)(Ch_x\dot{\rho}+mCh_t)\\ +Ch(mg_t-g_x\dot{\rho})+hh_{xt}-h_xh_t=0,\end{aligned}$$

$$\begin{aligned}2fg(mC^2\dot{\rho}+3ph)+g^2(mBC\dot{\rho}+pg)+2mAh(Cg-Bh)\dot{\rho}\\ +h_x(2Cf+Bg)\dot{\rho}+Bh(m(10Cf+Bg)-g_x)\dot{\rho}+m(2Cf+Bg)h_t\\ -mh(Bg_t+2Cf_t)+gh_{xt}+hg_{xt}-h_xg_t-h_tg_x=0,\end{aligned}$$

$$\begin{aligned}3fg(mBC\dot{\rho}+pg)+fh[4m(B^2+2AC)\dot{\rho}+3pf]-m(Cg+2Bh)f_t\\ +[(2Bf+Ag)\dot{\rho}-f_t]h_x+[m(2Bf+Ag)-f_x]h_t+(Cf-Ah)g_x\dot{\rho}\\ -(Cg+2Bh)f_x\dot{\rho}-mAh(Bg+2Ah)\dot{\rho}+fh_{xt}+gg_{xt}+hf_{xt}=0,\end{aligned}$$

$$\begin{aligned}fg(mB^2\dot{\rho}+3pf)-f_x(Bg+2Ah)\dot{\rho}+2mAg(Cf-Ah)\dot{\rho}\\ +mAB(6fh-g^2)\dot{\rho}+f(2Ah_x+Bg_x)\dot{\rho}-m(Bg+2Ah)f_t\\ +mf(Bg_t+2Ah_t)+fg_{xt}+gf_{xt}-f_tg_x-f_xg_t=0,\end{aligned}$$

$$\begin{aligned}mA(Bfg+2Afh-Ag^2)\dot{\rho}+Af(g_x\dot{\rho}+mg_t)\\ -Ag(f_x\dot{\rho}+mf_t)+ff_{xt}-f_xf_t+pf^{\,3}+q=0.\end{aligned}$$

where "dot" denotes d/dt and the subscript denotes the partial derivative with respect to the appropriate variable.

The previously mentioned system will be solved through the help of the symbolic Maple computation software, for some specific values or relationships existing between parameters, as follows:

(i) For $B=0$, $A=\pm C\neq 0$, $q=0$, we get the solutions:

$$\begin{aligned}g(t,x)=0,\ \rho(t)=\frac{\ln[-4A(c_1t+c_2)]}{4A},\\ f(t)=\pm\, h(t),\ h(t)=\frac{-2mC^2\dot{\rho}}{p},\end{aligned} \tag{17}$$

with m, c_1, c_2, p non-zero arbitrary constants.

(ii) For $A=B=0$, $C\neq 0$, $q=0$, there are two solutions:

$$f(t,x)=g(t,x)=0,\ h(t,x)=const=\frac{-2mc_3C^2}{p},\ \rho(t)=c_3t+c_4, \tag{18}$$

and

$$\begin{aligned} f(t,x) &= \frac{4c_6^2}{tc_6^2 - 2px + c_5c_6},\ g(t,x) = 0\ ,\\ h(t,x) &= -\left(\frac{mc_6C}{p}\right)^2,\ \rho(t) = \frac{mc_6^2}{2p}t + c_7, \end{aligned} \tag{19}$$

with m, c_3, c_4, c_5, c_6, c_7, p non-zero arbitrary constants.

(iii) For $B = 0$, $A = \pm C \neq 0$, $q \neq 0$, we obtain:

$$\begin{aligned} g(t,x) &= 0,\ \rho(t) = -\frac{\ln[-4A(c_8t + c_9)]}{4A},\\ f(t) &= \pm\frac{1}{3}\,h(t),\ h(t) = \frac{-3C^2}{2A^2}\left(\mp\frac{q}{p}\right)^{1/3}\dot{\rho}, \end{aligned} \tag{20}$$

with parameters q, p, c_8, c_9 different from zero.

(iv) For $A = 1$, $B = C = 2$, all the coefficient functions from (16) become constants, more exactly:

$$\begin{aligned} h(t) = g(t) \equiv h = const. &\in \left\{\frac{6(qp^2)^{1/3}}{p},\ \frac{-3(qp^2)^{1/3}}{p} \mp \frac{3\sqrt{3}(qp^2)^{1/3}}{p}i\right\},\\ f(t) \equiv f &= const = \frac{72q}{ph^2}, \end{aligned} \tag{21}$$

while the function $\rho(t)$ from $\xi = mx + \rho(t)$ takes a linear form:

$$\rho(t) = -\frac{ph}{8m}t + c_{10},\ c_{10} = const. \tag{22}$$

(v) For $A = 1$, $C = 2$, $B = -2$ a trivial solution is obtained, because

$$\begin{aligned} h(t,x) &= g(t,x) = 0,\\ f(t,x) = const. &\in \left\{\frac{(-qp^2)^{1/3}}{p},\ -\frac{(-qp^2)^{1/3}}{2p} \pm \frac{\sqrt{3}(-qp^2)^{1/3}}{2p}i\right\}. \end{aligned} \tag{23}$$

Remark 1: The situation $A = -1$, $B = \pm C$, $C = -2$ is omitted here, since it generate similar results to the ones provided by $A = 1$, $B = \pm C$, $C = 2$.

3.3. MULTIPLE SOLUTIONS FOR LIOUVILLE EQUATION

The former two situation (i) and (ii) from the previous study do lead to a rich variety of soliton-like solutions, periodic solutions and rational-type solutions for Liouville equation ($q = 0$ in (1) or (15)). More precisely, by substituting (17) into (16), we could generate periodic solutions and soliton-like solutions under the generical form:

$$v(t,x) = \mp \frac{mc_1C^2}{2pA(c_1t+c_2)} \left[1 \pm \varphi^2 \left(\xi(t,x)\right)\right]. \tag{24}$$

On one hand, multiple periodic solutions could be highlighted by choosing the upper sign in (24) and whatever $\varphi(\xi)$ from (6). On the other hand, a large variety of soliton-like solutions could be pointed out by choosing the lower sign in (24) and whatever hyperbolic functions $\varphi(\xi)$ from (5). In both situations we get $\xi(t,x) = mx + \frac{\ln[-4A(c_1t+c_2)]}{4A}$.

Furthermore, when substituting (18) or (19) into (16), we could could reach for rational-type solutions of Liouville equation, admitting the expressions:

$$v(t,x) = \frac{-2mc_3C^2}{p\left[C(mx+c_3t+c_4)+c_0\right]^2}, \tag{25}$$

and respectively

$$v(t,x) = \frac{4c_6^2}{tc_6^2 - 2px + c_5c_6} - \left(\frac{c_6mC}{p}\right)^2 \left[C\left(mx + \frac{mc_6^2}{2p}t + c_7\right) + c_0\right]^{-2}. \tag{26}$$

Remark 2: The variety of solutions $u(t,x)$ of Liouville equation (Eq. (1) with $q=0$) should be easily obtained from the solutions (24), (25) and (26) by making use of the transformation $u(t,x) = \ln v(t,x)$.

3.4. MULTIPLE SOLUTIONS FOR THE DBM EQUATION

The possibilities (iii) and (iv) above presented, could reveal for the DBM equation (15) with $q \neq 0$, various periodic solutions as well as soliton-type ones. Some of them may be written under the generical forms:

$$v(t,x) = \pm \frac{-3c_8C^2}{8A^3} \left(\mp \frac{q}{p}\right)^{1/3} \left[1 \pm 3\varphi^2 \left(\xi(t,x)\right)\right]. \tag{27}$$

Multiple periodic solutions should be derived by choosing the upper sign in (27) and whatever $\varphi(\xi)$ from (6) with $\xi(t,x) = \frac{3p^{2/3}(-q)^{1/3}}{4A^2}x - \frac{\ln[-4A(c_8t+c_9)]}{4A}$. A large variety of soliton-type solutions does appear when choosing the lower sign in (27) and whatever $\varphi(\xi)$ from (5) with $\xi(t,x) = \frac{3p^{2/3}(q)^{1/3}}{4A^2}x - \frac{\ln[-4A(c_8t+c_9)]}{4A}$. Other tan-type travelling wave solutions should appear under the general expressions:

$$v(t,x) = \frac{72q}{ph^2} + h\left[\varphi(\xi) + \varphi^2(\xi)\right], \tag{28}$$

with the constant h from (21) and the Riccati solutions $\varphi(\xi)$ from (7) with $\xi(t,x) = mx - \frac{ph}{8m}t + c_{10}$.

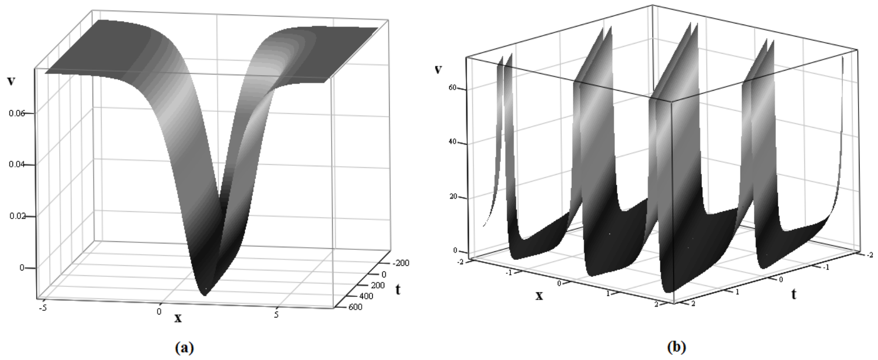

Fig. 1 – The graphical representations corresponding to: (a) soliton solution (27), where $\varphi(\xi) = \tanh(\xi)$ and $A = -C = 1$, $p = 2$, $q = 1/4$, $c_8 = -1/20$, $c_9 = -20$; (b) periodic solution (28), where $\varphi(\xi) = \tan(\xi)/(1+\tan(\xi))$ and $A = 1$, $C = 2$, $p = c_{10} = 5$, $q = m = 2$.

A soliton solution of type (27) and a periodic solution of type (28) are displayed in Fig. 1. The first one represents a smooth solitary wave and the second one a periodic wave of blow-up type. This last representation corresponds to a Riccati solution of the form:

$$\varphi(\xi) = \frac{\tan\xi}{1+\tan\xi}. \tag{29}$$

The associated DBM solution $u(t,x) = \ln v(t,x)$, with $v(t,x)$ given by (28) represents one of the new exact solutions which has been generated by our approach.

Remark 3: Other cot-type travellling wave solutions should be generated by taking into consideration the solutions $\varphi(\xi)$ from (8). The variety of solutions $u(t,x)$ of the DBM Eq. (1) could be directly obtained from the solutions (27) and (28) by making use of the transformation $u(t,x) = \ln v(t,x)$.

4. CONCLUDING REMARKS

We have presented an improved tanh-function method by considering some extensions to the traditional approach: (i) the dependence of the solutions for Riccati equation by three parameters instead of one; (ii) the wave variable $\xi = mx + \rho(t)$ is not compulsory linear in t; (iii) the solutions of the nonlinear ODEs are expressed in terms of Riccati solutions through finite series with variable coefficients. We have made use of this improved method in order to solve the special nonlinear DBM equation and its particular form represented by the Liouville equation. We got a large variety of both solitary waves and periodic solutions. For example, the solution (27) includes solitary waves for the lower sign and periodic solutions for the upper sign. The time dependency, expressed by $\rho(t)$, is not anymore linear, but logarithmic.

There is a larger class of DBM solutions, given by the multiple possible forms of Ricatti solutions $\varphi(\xi)$, given now by (6), (5), (7). Some of them, as for example that generated by (28) with $\varphi(\xi)$ expressed in (29) are not reported before in the study of DBM equation. We displayed in Fig. 1 two particular cases, corresponding to a smooth solitary wave and, respectively, to a periodic wave of blow-up type.

In contrast to other tanh-function method, some merits are to be considered in regard to the present method. Firstly, all the NPDEs which could be solved through other tanh-function methods may be as well easily solved through this method, and it generates more multiple soliton and periodic solutions. Secondly, we have made use here only of the special solutions of Riccati Eq. (4). If we make use of some other solutions of Eq. (4), we are able to generate more travelling wave solutions. Thirdly, this method, employing the symbolic computation' technique, has proved itself to be both simple and efficient. Therefore, it is readily applicable to a large variety of NPDEs.

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