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NEW RHEOLOGICAL MODELS WITHIN LOCAL FRACTIONAL DERIVATIVE

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Abstract. In this article, two local fractional rheological models *via* spring and dashpot elements in the one-dimensional case are proposed for the first time. The creep and relaxation behaviors of the fractal Maxwell fluid and Kelvin-Voigt solid are discussed. The constitutive relationships of the fractal mechanical elements are formulated by using local fractional calculus. The viscoelastic characteristics of the real materials are efficiently demonstrated and illustrated with the fractal charts.

Key words: rheology, fractal Maxwell fluid, fractal Kelvin-Voigt solid, local fractional derivative.

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1. INTRODUCTION

Fractional calculus (FC) has been successfully used in recent years in order to model the mathematical problems in real materials (see, for details, Refs. [1–12] and the works cited therein). For example, the representation of the constitutive relationship (RCR) of the fractional mechanical element with fractional derivative (FD) of the Riemann-Liouville type was given in [2]; its generalized version was proposed in [13]. The RCR with FC of the Liouville-Caputo type was considered in [14, 15]. The RCR *via* fractional derivative of the Caputo-Fabrizio type was presented in [16]. The historical perspective in the fractional linear viscoelasticity was reported for example in Refs. [17–21].

Recently, local fractional calculus (LFC) has played important roles in the description of the fractal diffusion [22], fluid flow [23], shallow water surfaces [24], free damped vibrations [25] and other areas (see, for example, [26–35] and related references cited therein). To date, LFC has not been utilized to consider the representations of the rheological behaviors in real materials. Motivated by the previous idea, from mathematical viewpoint, the chief objective of the present paper is to propose the local fractional rheological models (LFRMs) for real materials *via* fractal spring and dashpot elements involving local fractional derivative (LFD) and to investigate the creep and relaxation behaviors of the fractal Maxwell fluid and Kelvin-Voigt solid with the aid of the local fractional Laplace transform (LFLT).

This article is structured as follows. In Sec. 2, the LFD and LFLT of the non-differentiable function are formulated. In Sec. 3, the fractal spring and dashpot elements within LFD are proposed. In Sec. 4, the fractal Maxwell element *via* LFD is analyzed. The Kelvin-Voigt element *via* LFD is reported in Sec. 5. Finally, in Sec. 6, the conclusion is outlined briefly.

2. MATHEMATICAL FUNDAMENTALS

In this Section, the concepts of the LFD and LFLT of the non-differentiable functions, which are utilized in this paper, are presented.

The LFD of the non-differentiable function $\Lambda_\vartheta(t)$ of fractal order ϑ $(0<\vartheta<1)$ at the point $t=t_0$ is defined as follows (see, for details, [22–27, 31–35]):

$$\frac{\partial^\vartheta \Lambda_\vartheta(t_0)}{\partial t^\vartheta} = \lim_{t\to t_0} \frac{\Delta^\vartheta\left(\Lambda_\vartheta(t)-\Lambda_\vartheta(t_0)\right)}{(t-t_0)^\vartheta}, \tag{1}$$

where

$$\Delta^\vartheta\left(\Lambda_\vartheta(t)-\Lambda_\vartheta(t_0)\right) \cong \Gamma(1+\vartheta)\,\Delta\left[\Lambda_\vartheta(t)-\Lambda_\vartheta(t_0)\right].$$

The LFDs of the non-differentiable functions (see, for example, [22, 25, 26]) are listed in Table 1 below.

Table 1

The LFDs of the non-differentiable functions.

Non-differentiable functions	LFDs
$t^{k\vartheta}/\Gamma(1+k\vartheta)\ \ (k\in\mathrm{N})$	$t^{(k-1)\vartheta}/\Gamma[1+(k-1)\vartheta]$
$\mathrm{M}_\vartheta\left(\kappa t^\vartheta\right)$	$\kappa\mathrm{M}_\vartheta\left(\kappa t^\vartheta\right)$

The LFLT of the non-differentiable function $\psi_\vartheta(t)$ is given by [26]:

$$\psi_\vartheta(s) = {}_0I_\infty^{(\vartheta)}\left[\mathrm{M}_\vartheta\left(-s^\vartheta t^\vartheta\right)\psi_\vartheta(t)\right] \qquad (0<\vartheta\le 1), \tag{2}$$

where ${}_0I_\infty^{(\vartheta)}$ is the local fractional integral operator (see, for details, [24, 26]) and s

is the parameter of the local fractional Laplace integral operator (see [26]).

The LFLTs of the non-differentiable functions (see, *e.g.*, [26]) are listed in Table 2.

Table 2

The LFLTs of the non-differentiable functions

Non-differentiable functions	LFLTs
$t^{k\vartheta}/\Gamma(1+k\vartheta)\ \ (k \in \mathrm{N})$	$1/s^{(k+1)\vartheta}$
1	$1/s^{\vartheta}$
$\mathrm{M}_{\vartheta}\left(\kappa t^{\vartheta}\right)$	$1/\left(s^{\vartheta}-\kappa\right)$

3. FRACTAL SPRING AND DASHPOT ELEMENTS OF THE NON-DIFFERENTIABILITY TYPE

The representations of the fractal mechanical elements including the fractal spring and dashpot of the non-differentiability type are given in this Section.

As a matter of fact, the fractal spring of the non-differentiability type is the fractal elastic (or fractal storage) element illustrated in Fig. 1. This linear model, when represented as a perfect fractal elastic solid, obeys the Hooke's law of the non-differentiability type:

$$\sigma_{\vartheta}(t) = E_{\vartheta}\varepsilon_{\vartheta}(t), \tag{3}$$

where E_{ϑ} is the Young's modulus of the fractal material, $\varepsilon_{\vartheta}(t)$ is the fractal strain, and $\sigma_{\vartheta}(t)$ is the fractal stress.

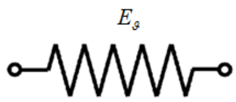

Fig. 1 – The representation of the fractal spring element.

The fractal dashpot within LFD is the fractal viscous (or fractal dissipative) element depicted in Fig. 2. The constitutive equation of non-differentiability type used to describe the viscous flow, which is expressed as the Newton's law of non-differentiability type, is given by:

$$\sigma_{\vartheta}(t) = \eta_{\vartheta}\frac{d^{\vartheta}\varepsilon_{\vartheta}(t)}{dt^{\vartheta}}, \tag{4}$$

where η_ϑ is the fractal viscosity, $\varepsilon_\vartheta(t)$ is the fractal strain, and $\sigma_\vartheta(t)$ is the fractal stress.

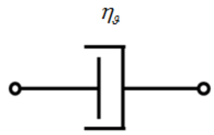

Fig. 2 – The representation of the fractal dashpot element within LFD.

With the aids of the Boltzmann superposition principle and causal histories for $t \in (0,\infty)$, the creep and the relaxation representations of the non-differentiability type are expressed by

$$\varepsilon_\vartheta(t) = \sigma_\vartheta\left(0^+\right) J_\vartheta(t) + \frac{1}{\Gamma(1+\vartheta)} \int_0^t J_\vartheta(t-\tau) \frac{d^\vartheta \sigma_\vartheta(\tau)}{d\tau^\vartheta} (d\tau)^\vartheta \tag{5}$$

and

$$\sigma_\vartheta(t) = \varepsilon_\vartheta\left(0^+\right) G_\vartheta(t) + \frac{1}{\Gamma(1+\vartheta)} \int_0^t G_\vartheta(t-\tau) \frac{d^\vartheta \varepsilon_\vartheta(\tau)}{d\tau^\vartheta} (d\tau)^\vartheta, \tag{6}$$

respectively.

From Eq. (5), the fractal creep compliance of the non-differentiability type is defined as follows:

$$J_\vartheta(t) = \frac{\varepsilon_\vartheta(t)}{\sigma_0}, \tag{7}$$

where $\sigma_\vartheta\left(0^+\right) = \sigma_0$ is the initial stress condition.

From Eq. (6), the fractal relaxation modulus of the non-differentiability type is defined by:

$$G_\vartheta(t) = \frac{\sigma_\vartheta(t)}{\varepsilon_0}, \tag{8}$$

where $\varepsilon_\vartheta\left(0^+\right) = \varepsilon_0$ is the initial strain condition.

Applying the LFLT to Eq. (5) and Eq. (6), we have

$$\varepsilon_\vartheta(s) = \frac{1}{s^\vartheta} J_\vartheta(s) \sigma_\vartheta(s) \tag{9}$$

and

$$\sigma_\vartheta(s) = \frac{1}{s^\vartheta} G_\vartheta(s) \varepsilon_\vartheta(s), \tag{10}$$

which lead us to

$$G_\vartheta(s) J_\vartheta(s) = \frac{1}{s^{2\vartheta}}, \tag{11}$$

where $\sigma_\vartheta(s)$, $\varepsilon_\vartheta(s)$, $G_\vartheta(s)$ and $J_\vartheta(s)$ are the LFLTs of $\sigma_\vartheta(t)$, $\varepsilon_\vartheta(t)$, $G_\vartheta(t)$, and $J_\vartheta(t)$, respectively.

From Eq. (11), the relationship between the fractal creep and the relaxation functions is of the following form:

$$\frac{1}{\Gamma(1+\vartheta)}\int_0^t G_\vartheta(t-\tau)\,J_\vartheta(\tau)\,(d\tau)^\vartheta = \frac{t^\vartheta}{\Gamma(1+\vartheta)} \tag{12}$$

or

$$\frac{1}{\Gamma(1+\vartheta)}\int_0^t J_\vartheta(t-\tau)\,G_\vartheta(\tau)\,(d\tau)^\vartheta = \frac{t^\vartheta}{\Gamma(1+\vartheta)}. \tag{13}$$

4. FRACTAL MAXWELL ELEMENT *VIA* LFD

The fractal Maxwell element, when adopted to represent the fractal fluid, is made up of the fractal spring and dashpot in series as illustrated in Fig. 3.

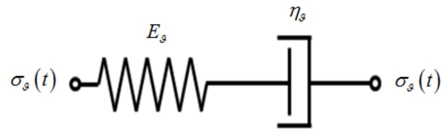

Fig. 3 – The representation of the fractal Maxwell fluid involving LFD.

The constitutive equation of the fractal Maxwell model within LFD can be written in the form:

$$\frac{d^\vartheta \varepsilon_\vartheta(t)}{dt^\vartheta} = \frac{\sigma_\vartheta(t)}{\eta_\vartheta} + \frac{1}{E_\vartheta}\frac{d^\vartheta \sigma_\vartheta(t)}{dt^\vartheta}, \tag{14}$$

where η_ϑ is the fractal viscosity, E_ϑ is the Young's modulus, $\varepsilon_\vartheta(t)$ is the fractal strain, and $\sigma_\vartheta(t)$ is the fractal stress.

When the fractal body is subjected to the following condition:

$$\sigma_\vartheta(t) = \sigma_0 H(t), \tag{15}$$

where $H(t)$ is the unit step function defined on fractal sets (see [26]) and σ_0 is the a constant stress, Eq. (14) becomes

$$\frac{d^\vartheta \varepsilon_\vartheta(t)}{dt^\vartheta} = \frac{\sigma_0(t)\,H(t)}{\eta_\vartheta}. \tag{16}$$

Upon taking the LFLT of both sides of Eq. (16), we have

$$s^{\vartheta}\varepsilon_{\vartheta}(s)-\frac{\sigma_0}{E_{\vartheta}}=\frac{\sigma_0}{s^{\vartheta}\eta_{\vartheta}}, \tag{17}$$

which reduces to

$$\varepsilon_{\vartheta}(t)=\sigma_0\left(\frac{1}{E_{\vartheta}}+\frac{1}{\eta_{\vartheta}}\frac{t^{\vartheta}}{\Gamma(1+\vartheta)}\right). \tag{18}$$

From Eq. (7), the fractal creep compliance of the non-differentiability type reads as follows:

$$J_{\vartheta}(t)=\frac{1}{E_{\vartheta}}+\frac{1}{\eta_{\vartheta}}\frac{t^{\vartheta}}{\Gamma(1+\vartheta)} \tag{19}$$

and its creep response is represented in Fig. 4.

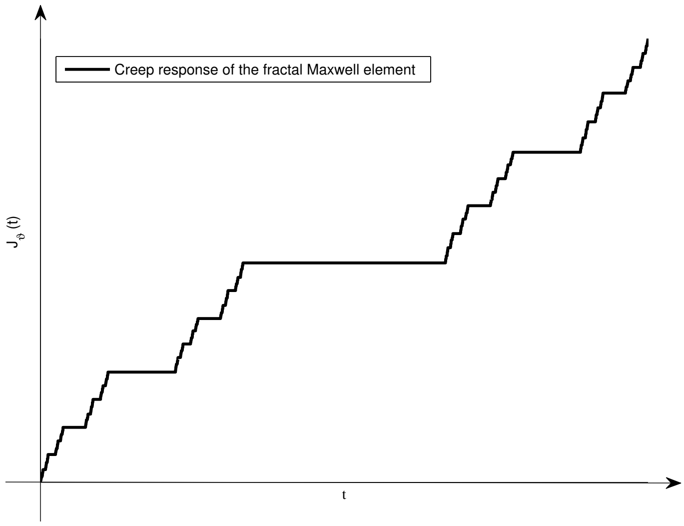

Fig. 4 – The creep response of the fractal Maxwell element for $\vartheta \ln 2/\ln 3$.

When the fractal body is subjected to the following condition:

$$\varepsilon_{\vartheta}(t)=\varepsilon_0 H(t), \tag{20}$$

where ε_0 is the a constant strain, Eq. (14) can be written in the form:

$$\frac{\sigma_{\vartheta}(t)}{\eta_{\vartheta}}+\frac{1}{E_{\vartheta}}\frac{d^{\vartheta}\sigma_{\vartheta}(t)}{dt^{\vartheta}}=0. \tag{21}$$

Taking the LFLT of both sides of Eq. (19), we have

$$\frac{\sigma_{\vartheta}(s)}{\eta_{\vartheta}}+\frac{1}{E_{\vartheta}}\left(s^{\vartheta}\sigma_{\vartheta}(s)-E_{\vartheta}\varepsilon_0\right)=0, \tag{22}$$

which leads us to

$$\sigma_\vartheta(s) = \frac{E_\vartheta \varepsilon_0}{s^\vartheta + \frac{E_\vartheta}{\eta_\vartheta}}. \tag{23}$$

For $\sigma_0 = E_\vartheta \varepsilon_0$, we find from Eq. (23) that

$$\sigma_\vartheta(t) = \sigma_0 \mathrm{M}_\vartheta \left(-\frac{E_\vartheta}{\eta_\vartheta} t^\vartheta \right). \tag{24}$$

In view of Eq. (49), the fractal relaxation modulus of the non-differentiability type can be written as follows:

$$G_\vartheta(t) = E_\vartheta \mathrm{M}_\vartheta \left(-\frac{t^\vartheta}{\tau_m} \right), \tag{25}$$

where τ_m given by

$$\tau_m = \eta_\vartheta / E_\vartheta$$

is the relaxation time, and its relaxation response is displayed in Fig. 5.

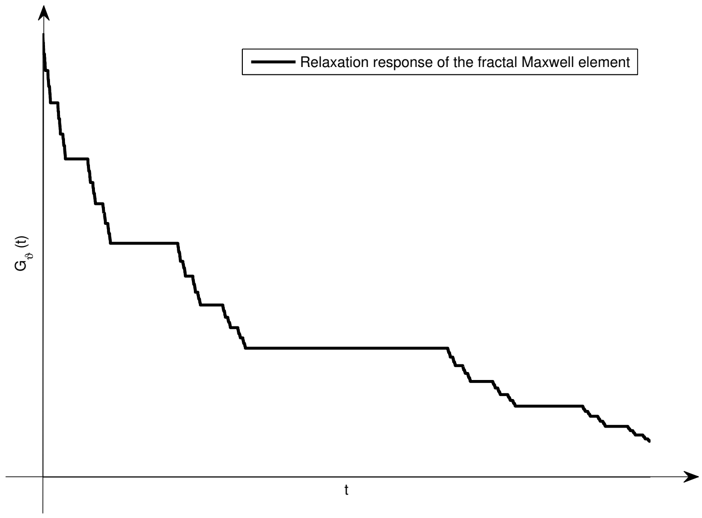

Fig. 5 – The relaxation response of the fractal Maxwell element for $\vartheta = \ln 2/\ln 3$.

5. FRACTAL KELVIN-VOIGT ELEMENT *VIA* LFD

The fractal Kelvin-Voigt element, which is used to represent the fractal solid, consists of the fractal spring and dashpot in parallel as illustrated in Fig. 6.

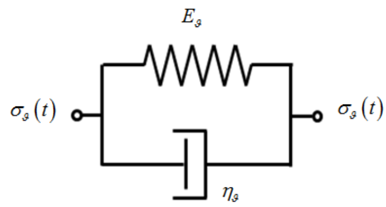

Fig. 6 – The representation of the fractal Kelvin-Voigt element involving LFD.

The constitutive equation of the fractal Kelvin-Voigt element within LFD takes the following form:

$$\eta_\vartheta \frac{d^\vartheta \varepsilon_\vartheta(t)}{dt^\vartheta} + E_\vartheta \varepsilon_\vartheta(t) = \sigma_\vartheta(t), \tag{26}$$

whereη_ϑ is the fractal viscosity, E_ϑ is the Young's modulus,$\varepsilon_\vartheta(t)$ is the fractal strain, and $\sigma_\vartheta(t)$ is the fractal stress.

When the fractal body is subjected to the following condition:

$$\sigma_\vartheta(t) = \sigma_0 H(t), \tag{27}$$

where σ_0 is the a constant stress, we deduce from Eq. (27) that

$$\eta_\vartheta \frac{d^\vartheta \varepsilon_\vartheta(t)}{dt^\vartheta} + E_\vartheta \varepsilon_\vartheta(t) = \sigma_0 H(t). \tag{28}$$

Thus, in terms of the LFLT of Eq. (27), we have

$$\eta_\vartheta \left(s^\vartheta \varepsilon_\vartheta(s) - \frac{\sigma_0}{E_\vartheta} \right) + E_\vartheta \varepsilon_\vartheta(s) = \frac{\sigma_0}{s^\vartheta}, \tag{29}$$

which implies that

$$\varepsilon_\vartheta(s) = \frac{\sigma_0}{E_\vartheta} \left(\frac{1}{s^\vartheta} - \frac{1}{s^\vartheta + \frac{E_\vartheta}{\eta_\vartheta}} \right). \tag{30}$$

From Eq. (30), the fractal strain of the solid is written in the following form:

$$\varepsilon_\vartheta(t) = \frac{\sigma_0}{E_\vartheta} \left(1 - M_\vartheta \left(-\frac{t^\vartheta}{\tau_K} \right) \right), \tag{31}$$

where $\tau_K = E_\vartheta / \eta_\vartheta$ is the retardation time.

In view of Eq. (30), the creep compliance of the fractal Kelvin-Voigt element can be written as follows:

$$J_\vartheta(t) = \frac{1}{E_\vartheta} \left(1 - M_\vartheta \left(-\frac{t^\vartheta}{\tau_K} \right) \right) \tag{32}$$

and its creep response is represented in Fig. 7.

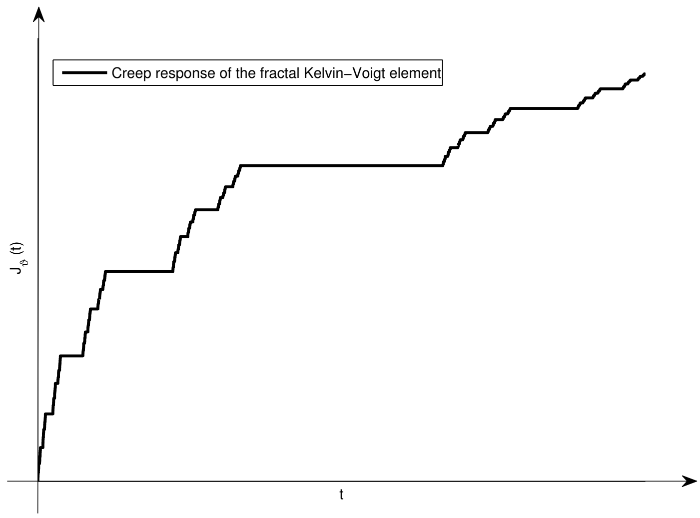

Fig. 7 – The creep response of the fractal Kelvin-Voigt element for $\vartheta = \ln 2/\ln 3$.

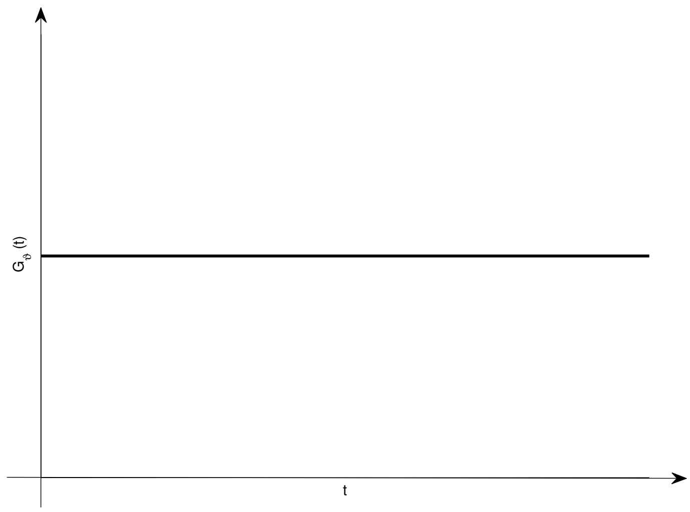

Fig. 8 – The relaxation response of the fractal Kelvin-Voigt element for $\vartheta = \ln 2/\ln 3$.

When the fractal body is subjected to the following condition:

$$\varepsilon_\vartheta (t) = \varepsilon_0 H (t), \tag{33}$$

where ε_0 is the a constant strain, we find from Eq. (26) that

$$\sigma_\vartheta(t) = E_\vartheta \varepsilon_0 + \eta_\vartheta \varepsilon_0 \frac{d^\vartheta H(t)}{dt^\vartheta}, \tag{34}$$

which yields the relaxation modulus of the fractal Kelvin-Voigt element as follows:

$$G_\vartheta(t) = E_\vartheta + \eta_\vartheta \frac{d^\vartheta H(t)}{dt^\vartheta}, \tag{35}$$

where $d^\vartheta H(t)/dt^\vartheta$ is the delta function defined on fractal sets [26], and its relaxation response is displayed in Fig. 8.

6. CONCLUSION

In our present investigation, we firstly addressed the LFRMs for the fractal Maxwell fluid and Kelvin-Voigt solid *via* spring and dashpot elements of non-differentiability type in the one-dimensional case based on the theory of LFC. The fractal creep compliance and relaxation modulus of the fractal Maxwell and Kelvin-Voigt elements were graphically presented with the aid of the LFLT. The viscoelastic behaviors of non-differentiability type in the fractal material were also discussed. For $\vartheta = 1$, the proposed models turn out to be those of the classical rheological behaviors (see [36] for further details of these ideas and concepts). Our presentation here may open up a new perspective in description of the fractal rheological behaviors of the real materials.

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