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ANOMALOUS DIFFUSION MODELS WITH GENERAL FRACTIONAL DERIVATIVES WITHIN THE KERNELS OF THE EXTENDED MITTAG-LEFFLER TYPE FUNCTIONS

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Abstract. This paper addresses the new general fractional derivatives (GFDs) involving the kernels of the extended Mittag-Leffler type functions (MLFs). With the aid of the GFDs in the MLF kernels, the mathematical models for the anomalous diffusion of fractional order are analyzed and discussed. The proposed formulations are also used to describe complex phenomena that occur in heat transfer.

Key words: General fractional derivative, Mittag-Leffler-function, anomalous diffusion, Laplace transform.

1. INTRODUCTION

Fractional derivatives (FDs) in the sense of the Liouville-Caputo and Riemann-Liouville definitions [1–9] have played an important role in applied and engineering sciences, such as in economy [10], mathematical biology [11], geosciences [12] and physics [13–15]. The general fractional derivatives (GFDs) were considered in [16–19] and other formulations in the Riemann-Liouville sense were proposed in [20, 21]. Recently, a fractional derivative in Caputo-Liouville sense involving the exponential function was suggested in [22] and further discussed in [23–25]. Moreover, the generalized versions of the above FDs were proposed in [26]. They were successfully adopted in the anomalous diffusion models in complex media [21, 25].

The Mittag-Leffler function (MLF) was first proposed by Swedish mathematician Gosta Mittag-Leffler in 1903 [27]. The extended MLFs were formulated later.

For example, the two-parametric MLF was reported by Wiman in 1905 [28]. The three-parametric MLF was considered by Prabhakar in 1971 [29]. The four-parametric MLF was proposed more recently by the Shukla *et al.* [30], and Shivastava *et al.* [31]. The multiple MLFs were presented in [32–34]. The Liouville-Caputo and Riemann-Liouville FDs involving the normalized process were proposed with a negative-parametric MLF kernel (in the MLF kernel there exist a constant from the normalized process) in [24], and the variable-order FDs in the Liouville-Caputo sense were further developed in [26]. The generalized versions of the GFDs in the extended MLF kernels were reported in [27–38]. The extended MLFs were adopted in the different areas of the mathematical physics and mechanics (see [35–38]).

Motivated by studies of physical phenomena with complex behaviors following exponential laws (see, for example, [35]), this paper proposes the Liouville-Caputo and Riemann-Liouville types GFDs in kernels of the extended MLFs and discusses the anomalous diffusion models of fractional order.

The paper is organized as follows. Section 2 introduces the extended MLFs and the concept of GFDs. Section 3 analyzes the anomalous diffusion models of fractional order and their solutions. Finally, section 4 outlines the main conclusions.

2. GFDS IN THE KERNELS OF EXTENDED MLFS

In this section, we will present the extended MLFs and propose the *general fractional calculus* (GFC) in the extended MLFs kernels.

2.1. EXTENDED MLFS

Suppose that $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}_0^+$, $\mathbb{N}$, and $\mathbb{N}_0$ are the sets of complex numbers, real numbers, non-negative real numbers, positive integers, and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, respectively.

The MLF is defined by [27]:

$$E_\nu(\eta) = \sum_{\kappa=0}^{\infty} \frac{\eta^\kappa}{\Gamma(\kappa\nu+1)}, \tag{1}$$

where $\eta, \nu \in \mathbb{C}$, $\Re(\nu) \in \mathbb{R}_0^+$, $\kappa \in \mathbb{N}$, and $\Gamma(\cdot)$ is the Gamma function [3].

The extended two-parametrical MLF is defined by the expression [28]:

$$E_{\nu,\upsilon}(\eta) = \sum_{\kappa=0}^{\infty} \frac{\eta^\kappa}{\Gamma(\kappa\nu+\upsilon)}, \tag{2}$$

where $\eta, \nu, \upsilon \in \mathbb{C}$, $\Re(\nu), \Re(\upsilon) \in \mathbb{R}_0^+$, and $\kappa \in \mathbb{N}$.

The extended three-parametrical MLF is given by [29]:

$$E_{\nu,\upsilon}^{\varphi}(\eta) = \sum_{\kappa=0}^{\infty} \frac{(\varphi)_\kappa}{\Gamma(\kappa\nu+\upsilon)} \frac{\eta^\kappa}{\Gamma(\kappa+1)}, \tag{3}$$

where $\eta,\nu,\upsilon,\varphi\in\mathbb{C}$, $\Re(\nu),\Re(\upsilon),\Re(\varphi)\in\mathbb{R}_0^+$, $\kappa\in\mathbb{N}$, and the Pochhammer symbol is [2]:

$$(\varphi)_\kappa=\begin{cases}1,\ \kappa=0,\\ \frac{\Gamma(\varphi+\kappa)}{\Gamma(\varphi)},\kappa\in\mathbb{N}.\end{cases}\tag{4}$$

The extended four-parametrical MLF is given as follows [31]:

$$E_{\nu,\upsilon}^{\varphi;\phi}(\eta)=\sum_{\kappa=0}^{\infty}\frac{(\varphi)_{\kappa\phi}}{\Gamma(\kappa\nu+\upsilon)}\frac{\eta^\kappa}{\Gamma(\kappa+1)},\tag{5}$$

where $\eta,\nu,\upsilon,\varphi,\phi\in\mathbb{C}$, $\Re(\nu)>\max\{0,\Re(\phi)-1\},\Re(\phi)\in\mathbb{R}_0^+$, $\kappa\in\mathbb{N}$, with a special version that follows the conditions [30]:

$$\phi=m\,(m\in(0,1)\cup\mathbb{N})\text{ and }\min\{\Re(\upsilon),\Re(\varphi)\}>0.\tag{6}$$

An extended MLF is given by [32]:

$$E_{\varphi,\phi}((\nu_1,\upsilon_1),...,(\nu_n,\upsilon_n);\eta)=\sum_{\kappa=0}^{\infty}\frac{(\varphi)_{\kappa\phi}}{\prod_{j=1}^{n}\Gamma(\kappa\nu_j+\upsilon_j)}\frac{\eta^\kappa}{\Gamma(\kappa+1)},\tag{7}$$

where $\eta,\nu_j,\upsilon_j,\varphi,\phi\in\mathbb{C}$, $j,\kappa\in\mathbb{N}$, $\Re(\nu_j),\Re(\upsilon_j),\Re(\phi)\in\mathbb{R}_0^+$, and $\sum_{j=1}^{m}\Re(\nu_j)>\max\{0,\Re(\phi)-1\}$. The special cases for expression (7) are when [33]

$$\phi=m\,(m\in(0,1)\cup\mathbb{N})\text{ and }\sum_{j=1}^{m}\Re(\nu_j)>\max\{0,m-1\},\tag{8}$$

and when [34]

$$\varphi=1\text{ and }\phi=1.\tag{9}$$

The Laplace transforms of the extended MLFs are given as follows [33, 38]:

(T1)

$$\mathrm{L}[E_\nu(\varepsilon\eta^\nu)]=\frac{1}{s}{}_2\Psi_2\left[\begin{array}{cc}(\nu,1)(1,1); & \frac{\varepsilon}{s^\nu}\\ (\nu,1)(0,1); & \end{array}\right];\tag{10}$$

(T2)

$$\mathrm{L}[E_{\nu,\upsilon}(\varepsilon\eta^\nu)]=\frac{1}{s}{}_2\Psi_2\left[\begin{array}{cc}(\nu,1)(1,1); & \frac{\varepsilon}{s^\nu}\\ (\nu,\upsilon)(0,1); & \end{array}\right];\tag{11}$$

(T3)

$$\mathrm{L}\left[E_{\nu,\upsilon}^{\varphi}(\varepsilon\eta^\nu)\right]=\frac{1}{s}{}_2\Psi_2\left[\begin{array}{cc}(\nu,1)(1,\varphi); & \frac{\varepsilon}{s^\nu}\\ (\nu,\upsilon)(0,\varphi); & \end{array}\right];\tag{12}$$

(T4)

$$\mathrm{L}\left[E_{\nu,\upsilon}^{\varphi;\phi}(\varepsilon\eta^\nu)\right]=\frac{1}{s}{}_2\Psi_2\left[\begin{array}{cc}(\nu,1)(\phi,\varphi); & \frac{\varepsilon}{s^\nu}\\ (\nu,\upsilon)(0,\varphi); & \end{array}\right];\tag{13}$$

(T5)

$$\mathrm{L}\left[E_{\varphi,\phi}\left(\left(\nu_1,\upsilon_1\right),...,\left(\nu_n,\upsilon_n\right);\varepsilon\eta^{\nu}\right)\right]=\frac{1}{s}{}_2\Psi_{n+1}\left[\begin{array}{ll}(\nu,1)\,(\phi,\varphi); & \\ (\nu_1,\upsilon_1)\ldots(\nu_n,\upsilon_n)\,(0,\varphi); & \frac{\varepsilon}{s^{\nu}}\end{array}\right], \tag{14}$$

where ε is a constant, the generalized Wright function ${}_{\varpi}\Psi_{\omega}\left(\varpi,\omega\in\mathbb{N}_0\right)$ is denoted as [33]:

$${}_{\varpi}\Psi_{\omega}\left[\begin{array}{ll}(x_1,X_1),...,(x_{\varpi},X_{\varpi}); & \\ (y_1,Y_1),...,(y_{\omega},Y_{\omega}); & \eta\end{array}\right]=\sum_{\kappa=0}^{\infty}\frac{\prod_{j=0}^{\varpi}\Gamma(\kappa x_j+X_j)\cdots\Gamma(\kappa x_j+X_j)}{\prod_{j=0}^{\omega}\Gamma(\kappa y_j+Y_j)\cdots\Gamma(\kappa y_j+Y_j)} \times\frac{\eta^{\kappa}}{\Gamma(\kappa+1)}, \tag{15}$$

($\sum_{j=1}^{m}\Re\left(X_j\right)>0;\sum_{j=1}^{m}\Re\left(Y_j\right)>0;1+\left(\sum_{j=1}^{m}\Re\left(Y_j\right)-\sum_{j=1}^{m}\Re\left(X_j\right)\right)\geq 0$) and the Laplace transform is denoted by [33]:

$$\mathrm{L}\left[f\left(\eta\right)\right]:=\int_0^{\infty}e^{-s\eta}f\left(\eta\right)d\eta. \tag{16}$$

2.2. GFDS IN THE EXTENDED MLFS KERNELS

Here we address the GFDs in the extended MLFs types kernels.

The GFDs of the Liouville-Caputo and Riemann-Liouville types are defined by:

$$\left(\mathrm{D}_{(o)}^{C}\Omega\right)(\tau)=\int_a^{\tau}o(\tau-t)\,\Omega^{(1)}(t)\,dt\ \ (\tau>a), \tag{17}$$

$$\left(\mathrm{D}_{(o)}^{RL}\Omega\right)(\tau)=\frac{d}{d\tau}\int_a^{\tau}o(\tau-t)\,\Omega(t)\,dt\ \ (\tau>a), \tag{18}$$

respectively, where $a\in[-\infty,+\infty)$, $d\Omega(\tau)/d\tau=\Omega^{(1)}(t)$, $\Omega^{(1)}\in L_1^{loc}\left(\mathbb{R}_0^+\right)$, and $o(\tau)$ is the kernel function.

We obtain the relationship between Eqs. (17) and (18) as follows:

$$\left(\mathrm{D}_{(o)}^{C}\Omega\right)(\tau)=\left(\mathrm{D}_{(o)}^{RL}\Omega\right)(\tau)-o(\tau)\,\Omega(a). \tag{19}$$

For $a=0$, the GFDs of the Liouville-Caputo and Riemann-Liouville types in different kernel functions are discussed in [18, 19].

We now consider the extended MLFs as the kernel functions in Eqs. (17) and (18).

Employing the Saxena-Nishimoto type MLF, given as:

$$o(\tau)=E_{\varphi,\phi}\left(\left(\nu_1,\upsilon_1\right),...,\left(\nu_n,\upsilon_n\right);\tau^{\nu}\right), \tag{20}$$

the GFDs of the Liouville-Caputo and Riemann-Liouville types are defined as:

$$\left({}^{C}_{E_{\varphi,\phi}}\mathrm{D}_a^{(\nu)}\Omega\right)(\tau)=\int_a^\tau E_{\varphi,\phi}\left((\nu_1,\upsilon_1),...,(\nu_n,\upsilon_n);(\tau-t)^\nu\right)\Omega^{(1)}(t)\,dt\ \ (\tau>a), \tag{21}$$

$$\left({}^{RL}_{E_{\varphi,\phi}}\mathrm{D}_a^{(\nu)}\Omega\right)(\tau)=\frac{d}{d\tau}\int_a^\tau E_{\varphi,\phi}\left((\nu_1,\upsilon_1),...,(\nu_n,\upsilon_n);(\tau-t)^\nu\right)\Omega(t)\,dt\ \ (\tau>a), \tag{22}$$

respectively, where

$$\nu\in(0,1), a\in[0,+\infty),\ \text{and}\ \Omega^{(1)}\in L_1^{loc}\left(\mathbb{R}_0^+\right). \tag{23}$$

From Eq. (20) we have

$$\left({}^{C}_{E_{\varphi,\phi}}\mathrm{D}_a^{(\nu)}\Omega\right)(\tau)=\left({}^{RL}_{E_{\varphi,\phi}}\mathrm{D}_a^{(\nu)}\Omega\right)(\tau)-E_{\varphi,\phi}\left((\nu_1,\upsilon_1),...,(\nu_n,\upsilon_n);\tau^\nu\right)\Omega(a). \tag{24}$$

The GFDs of the Liouville-Caputo and Riemann-Liouville types in the negative-parametric Saxena-Nishimoto type MLF are defined as:

$$\left({}^{C}_{E_{\varphi,\phi}(-)}\mathrm{D}_a^{(\nu)}\Omega\right)(\tau)=\int_a^\tau E_{\varphi,\phi}\left((\nu_1,\upsilon_1),...,(\nu_n,\upsilon_n);-(\tau-t)^\nu\right)\Omega^{(1)}(t)\,dt\ \ (\tau>a), \tag{25}$$

$$\left({}^{RL}_{E_{\varphi,\phi}(-)}\mathrm{D}_a^{(\nu)}\Omega\right)(\tau)=\frac{d}{d\tau}\int_a^\tau E_{\varphi,\phi}\left((\nu_1,\upsilon_1),...,(\nu_n,\upsilon_n);-(\tau-t)^\nu\right)\Omega(t)\,dt\ \ (\tau>a), \tag{26}$$

respectively, where

$$\nu\in(0,1), a\in[0,+\infty),\ \text{and}\ \Omega^{(1)}\in L_1^{loc}\left(\mathbb{R}_0^+\right). \tag{27}$$

From Eq. (20) we obtain

$$\left({}^{C}_{E_{\varphi,\phi}(-)}\mathrm{D}_a^{(\nu)}\Omega\right)(\tau)=\left({}^{RL}_{E_{\varphi,\phi}(-)}\mathrm{D}_a^{(\nu)}\Omega\right)(\tau)-E_{\varphi,\phi}\left((\nu_1,\upsilon_1),...,(\nu_n,\upsilon_n);-\tau^\nu\right)\Omega(a). \tag{28}$$

Remark 1. We now give the special cases of Eqs. (21) and (22) as follows:
(M1) The GFDs of the Liouville-Caputo and Riemann-Liouville types in the kernel of the Shukla-Prajapati-Srivastava-Tomovski type MLF (5) are defined as:

$$\left({}^{C}_{E_{\nu,\upsilon}^{\varphi,\phi}}\mathrm{D}_a^{(\nu)}\Omega\right)(\tau)=\int_a^\tau E_{\nu,\upsilon}^{\varphi,\phi}\left((\tau-t)^\nu\right)\Omega^{(1)}(t)\,dt\ \ (\tau>a), \tag{29}$$

$$\left({}^{RL}_{E_{\nu,\upsilon}^{\varphi,\phi}}\mathrm{D}_a^{(\nu)}\Omega\right)(\tau)=\frac{d}{d\tau}\int_a^\tau E_{\nu,\upsilon}^{\varphi,\phi}\left((\tau-t)^\nu\right)\Omega(t)\,dt\ \ (\tau>a), \tag{30}$$

respectively, where

$$\left({}^{C}_{E_{\nu,\upsilon}^{\varphi,\phi}}\mathrm{D}_a^{(\nu)}\Omega\right)(\tau)=\left({}^{RL}_{E_{\nu,\upsilon}^{\varphi,\phi}}\mathrm{D}_a^{(\nu)}\Omega\right)(\tau)-E_{\nu,\upsilon}^{\varphi,\phi}\left(\tau^\nu\right)\Omega(a). \tag{31}$$

The GFDs of the Liouville-Caputo and Riemann-Liouville types in the kernel of the negative-parametric Shukla-Prajapati-Srivastava-Tomovski type MLF (5) are defined as:

$$\left({}^{C}_{E^{\varphi,\phi}_{\nu,\upsilon}(-)}\mathrm{D}^{(\nu)}_{a}\Omega\right)(\tau)=\int_a^\tau E^{\varphi,\phi}_{\nu,\upsilon}\left(-(\tau-t)^{\nu}\right)\Omega^{(1)}(t)\,dt\ (\tau>a), \tag{32}$$

$$\left({}^{RL}_{E^{\varphi,\phi}_{\nu,\upsilon}(-)}\mathrm{D}^{(\nu)}_{a}\Omega\right)(\tau)=\frac{d}{d\tau}\int_a^\tau E^{\varphi,\phi}_{\nu,\upsilon}\left(-(\tau-t)^{\nu}\right)\Omega(t)\,dt\ (\tau>a), \tag{33}$$

respectively, where

$$\left({}^{C}_{E^{\varphi,\phi}_{\nu,\upsilon}(-)}\mathrm{D}^{(\nu)}_{a}\Omega\right)(\tau)=\left({}^{RL}_{E^{\varphi,\phi}_{\nu,\upsilon}(-)}\mathrm{D}^{(\nu)}_{a}\Omega\right)(\tau)-E^{\varphi,\phi}_{\nu,\upsilon}\left(-\tau^{\nu}\right)\Omega(a). \tag{34}$$

(M2) The GFDs of the Liouville-Caputo and Riemann-Liouville types in the kernel of the Prabhakar type MLF (3) are defined as:

$$\left({}^{C}_{E^{\varphi}_{\nu,\upsilon}}\mathrm{D}^{(\nu)}_{a}\Omega\right)(\tau)=\int_a^\tau E^{\varphi}_{\nu,\upsilon}\left((\tau-t)^{\nu}\right)\Omega^{(1)}(t)\,dt\ (\tau>a), \tag{35}$$

$$\left({}^{RL}_{E^{\varphi}_{\nu,\upsilon}}\mathrm{D}^{(\nu)}_{a}\Omega\right)(\tau)=\frac{d}{d\tau}\int_a^\tau E^{\varphi}_{\nu,\upsilon}\left((\tau-t)^{\nu}\right)\Omega(t)\,dt\ (\tau>a), \tag{36}$$

respectively, where

$$\left({}^{C}_{E^{\varphi}_{\nu,\upsilon}}\mathrm{D}^{(\nu)}_{a}\Omega\right)(\tau)=\left({}^{RL}_{E^{\varphi}_{\nu,\upsilon}}\mathrm{D}^{(\nu)}_{a}\Omega\right)(\tau)-E^{\varphi}_{\nu,\upsilon}\left(\tau^{\nu}\right)\Omega(a). \tag{37}$$

The GFDs of the Liouville-Caputo and Riemann-Liouville types in the kernel of the negative-parametric Prabhakar type MLF (3) are defined as:

$$\left({}^{C}_{E^{\varphi}_{\nu,\upsilon}(-)}\mathrm{D}^{(\nu)}_{a}\Omega\right)(\tau)=\int_a^\tau E^{\varphi}_{\nu,\upsilon}\left(-(\tau-t)^{\nu}\right)\Omega^{(1)}(t)\,dt\ (\tau>a), \tag{38}$$

$$\left({}^{RL}_{E^{\varphi}_{\nu,\upsilon}(-)}\mathrm{D}^{(\nu)}_{a}\Omega\right)(\tau)=\frac{d}{d\tau}\int_a^\tau E^{\varphi}_{\nu,\upsilon}\left(-(\tau-t)^{\nu}\right)\Omega(t)\,dt\ (\tau>a), \tag{39}$$

respectively, where

$$\left({}^{C}_{E^{\varphi}_{\nu,\upsilon}(-)}\mathrm{D}^{(\nu)}_{a}\Omega\right)(\tau)=\left({}^{RL}_{E^{\varphi}_{\nu,\upsilon}(-)}\mathrm{D}^{(\nu)}_{a}\Omega\right)(\tau)-E^{\varphi}_{\nu,\upsilon}\left(-\tau^{\nu}\right)\Omega(a). \tag{40}$$

(M3) The GFDs of the Liouville-Caputo and Riemann-Liouville types in the kernel of the Wiman type MLF (2) are defined as:

$$\left({}^{C}_{E_{\nu,\upsilon}}\mathrm{D}^{(\nu)}_{a}\Omega\right)(\tau)=\int_a^\tau E_{\nu,\upsilon}\left((\tau-t)^{\nu}\right)\Omega^{(1)}(t)\,dt\ (\tau>a), \tag{41}$$

$$\left({}^{RL}_{E_{\nu,\upsilon}}\mathrm{D}^{(\nu)}_{a}\Omega\right)(\tau)=\frac{d}{d\tau}\int_a^\tau E_{\nu,\upsilon}\left((\tau-t)^{\nu}\right)\Omega(t)\,dt\ (\tau>a), \tag{42}$$

respectively, where

$$\left({}^{C}_{E_{\nu,\upsilon}}\mathrm{D}^{(\nu)}_{a}\Omega\right)(\tau)=\left({}^{RL}_{E_{\nu,\upsilon}}\mathrm{D}^{(\nu)}_{a}\Omega\right)(\tau)-E_{\nu,\upsilon}\left(\tau^{\nu}\right)\Omega(a). \tag{43}$$

The GFDs of the Liouville-Caputo and Riemann-Liouville types in the kernel of the negative-parametric Wiman type MLF (2) are defined as:

$$\left({}_{E_{\nu,\upsilon}(-)}^{C}\mathrm{D}_a^{(\nu)}\Omega\right)(\tau)=\int_a^{\tau}E_{\nu,\upsilon}\left(-(\tau-t)^{\nu}\right)\Omega^{(1)}(t)\,dt\ \ (\tau>a), \tag{44}$$

$$\left({}_{E_{\nu,\upsilon}(-)}^{RL}\mathrm{D}_a^{(\nu)}\Omega\right)(\tau)=\frac{d}{d\tau}\int_a^{\tau}E_{\nu,\upsilon}\left(-(\tau-t)^{\nu}\right)\Omega(t)\,dt\ \ (\tau>a), \tag{45}$$

respectively, where

$$\left({}_{E_{\nu,\upsilon}(-)}^{C}\mathrm{D}_a^{(\nu)}\Omega\right)(\tau)=\left({}_{E_{\nu,\upsilon}(-)}^{RL}\mathrm{D}_a^{(\nu)}\Omega\right)(\tau)-E_{\nu,\upsilon}\left(-\tau^{\nu}\right)\Omega(a). \tag{46}$$

(M4) The GFDs of the Liouville-Caputo and Riemann-Liouville types in the kernel of the MLF (1) are defined as:

$$\left({}_{E_{\nu}}^{C}\mathrm{D}_a^{(\nu)}\Omega\right)(\tau)=\int_a^{\tau}E_{\nu}\left((\tau-t)^{\nu}\right)\Omega^{(1)}(t)\,dt\ \ (\tau>a), \tag{47}$$

$$\left({}_{E_{\nu}}^{RL}\mathrm{D}_a^{(\nu)}\Omega\right)(\tau)=\frac{d}{d\tau}\int_a^{\tau}E_{\nu}\left((\tau-t)^{\nu}\right)\Omega(t)\,dt\ \ (\tau>a), \tag{48}$$

respectively, where

$$\left({}_{E_{\nu}}^{C}\mathrm{D}_a^{(\nu)}\Omega\right)(\tau)=\left({}_{E_{\nu}}^{RL}\mathrm{D}_a^{(\nu)}\Omega\right)(\tau)-E_{\nu}\left(\tau^{\nu}\right)\Omega(a). \tag{49}$$

(M5) The GFDs of the Liouville-Caputo and Riemann-Liouville types in the kernel of the stretched exponential function are defined by:

$$\left({}_{exp_{\nu}}^{C}\mathrm{D}_a^{(\nu)}\Omega\right)(\tau)=\int_a^{\tau}exp_{\nu}\left((\tau-t)^{\nu}\right)\Omega^{(1)}(t)\,dt\ \ (\tau>a), \tag{50}$$

$$\left({}_{exp_{\nu}}^{RL}\mathrm{D}_a^{(\nu)}\Omega\right)(\tau)=\frac{d}{d\tau}\int_a^{\tau}exp_{\nu}\left((\tau-t)^{\nu}\right)\Omega(t)\,dt\ \ (\tau>a), \tag{51}$$

respectively, where

$$\left({}_{exp_{\nu}}^{C}\mathrm{D}_a^{(\nu)}\Omega\right)(\tau)=\left({}_{exp_{\nu}}^{RL}\mathrm{D}_a^{(\nu)}\Omega\right)(\tau)-exp_{\nu}\left(\tau^{\nu}\right)\Omega(a), \tag{52}$$

and the stretched exponential function (called the exponential type function) is [19]

$$exp_{\nu}\left(\tau^{\nu}\right)=\sum_{\kappa=0}^{\infty}\frac{\tau^{\kappa\nu}}{\Gamma(\kappa+1)}. \tag{53}$$

The GFDs of the Liouville-Caputo and Riemann-Liouville types in the kernel of the negative-parametric stretched exponential function are defined by:

$$\left({}_{exp_{\nu}(-)}^{C}\mathrm{D}_a^{(\nu)}\Omega\right)(\tau)=\int_a^{\tau}exp_{\nu}\left(-(\tau-t)^{\nu}\right)\Omega^{(1)}(t)\,dt\ \ (\tau>a), \tag{54}$$

$$\left({}^{RL}_{exp_\nu(-)}\mathrm{D}_a^{(\nu)}\Omega\right)(\tau)=\frac{d}{d\tau}\int_a^\tau exp_\nu\left(-(\tau-t)^\nu\right)\Omega(t)\,dt\ \ (\tau>a), \tag{55}$$

respectively, where

$$\left({}^{C}_{exp_\nu(-)}\mathrm{D}_a^{(\nu)}\Omega\right)(\tau)=\left({}^{RL}_{exp_\nu(-)}\mathrm{D}_a^{(\nu)}\Omega\right)(\tau)-exp_\nu\left(-\tau^\nu\right)\Omega(a). \tag{56}$$

(M6) The GFDs of the Liouville-Caputo and Riemann-Liouville types in the kernel of the exponential function are defined as:

$$\left({}^{C}_{exp}\mathrm{D}_a^{(1)}\Omega\right)(\tau)=\int_a^\tau exp(\tau-t)\,\Omega^{(1)}(t)\,dt\ \ (\tau>a), \tag{57}$$

$$\left({}^{RL}_{exp}\mathrm{D}_a^{(1)}\Omega\right)(\tau)=\frac{d}{d\tau}\int_a^\tau exp(\tau-t)\,\Omega(t)\,dt\ \ (\tau>a), \tag{58}$$

respectively, where

$$\left({}^{C}_{exp}\mathrm{D}_a^{(1)}\Omega\right)(\tau)=\left({}^{RL}_{exp}\mathrm{D}_a^{(1)}\Omega\right)(\tau)-exp(\tau)\,\Omega(a), \tag{59}$$

and the exponential function is given by [16, 17]:

$$exp(\tau)=\sum_{\kappa=0}^{\infty}\frac{\tau^\kappa}{\Gamma(\kappa+1)}. \tag{60}$$

Similarly, the corresponding general fractional integral is defined as:

$$\left({}_{exp}\mathrm{I}_0^{(1)}\Omega\right)(\tau)=\Omega(\tau)-\int_0^\tau(\tau-t)\,\Omega(t)\,dt\ \ (\tau>0). \tag{61}$$

Let $\tau\in\mathbb{R}_0^+$, $\nu\in(0,1)$, $\Xi\in L\left(\mathbb{R}_0^+\right)$ and $\Omega^{(1)}\in L_1^{loc}\left(\mathbb{R}_0^+\right)$. The Abel type integral equation of the second kind

$$\Xi(\tau)-\int_0^\tau(\tau-t)\,\Xi(t)\,dt=\Omega(\tau), \tag{62}$$

has a unique solution

$$\Xi(t)=\int_0^\tau exp(\tau-t)\,\Omega^{(1)}(t)\,dt-exp(\tau)\,\Omega(0), \tag{63}$$

where $\Omega(\tau=0)=\Omega(0)$.

The GFDs of the Liouville-Caputo and Riemann-Liouville types in the kernel of the negative-parametric exponential function are defined as:

$$\left({}^{C}_{exp(-)}\mathrm{D}_a^{(1)}\Omega\right)(\tau)=\int_a^\tau exp(-(\tau-t))\,\Omega^{(1)}(t)\,dt\ \ (\tau>a), \tag{64}$$

$$\left({}^{RL}_{exp(-)}\mathrm{D}_a^{(1)}\Omega\right)(\tau)=\frac{d}{d\tau}\int_a^\tau exp(-(\tau-t))\,\Omega(t)\,dt\ \ (\tau>a), \tag{65}$$

respectively, where

$$\left({}^{C}_{exp(-)}\mathrm{D}_a^{(1)}\Omega\right)(\tau)=\left({}^{RL}_{exp(-)}\mathrm{D}_a^{(1)}\Omega\right)(\tau)-exp\,(-\tau)\,\Omega\,(a)\,. \tag{66}$$

Similarly, the corresponding general fractional integral is defined as:

$$\left({}_{exp(-)}\mathrm{I}_0^{(\nu)}\Omega\right)(\tau)=\Omega\,(\tau)+\int_0^{\tau}(\tau-t)\,\Omega\,(t)\,dt\ \ (\tau>0)\,. \tag{67}$$

Let $\tau\in\mathbb{R}_0^+$, $\nu\in(0{,}1)$, $\Xi\in L\left(\mathbb{R}_0^+\right)$, and $\Omega^{(1)}\in L_1^{loc}\left(\mathbb{R}_0^+\right)$. The Abel type integral equation of the second kind

$$\Xi\,(\tau)+\int_0^{\tau}(\tau-t)\,\Xi\,(t)\,dt=\Omega\,(\tau)\,, \tag{68}$$

has a unique solution

$$\Xi\,(t)=\int_0^{\tau}exp\,(-(\tau-t))\,\Omega^{(1)}\,(t)\,dt-exp\,(-\tau)\,\Omega\,(0)\,, \tag{69}$$

where $\Omega\,(\tau=0)=\Omega\,(0)$.

Remark 2. According to the ideas discussed in [16], the general fractional integral, that has the relationship with Eqs. (47) and (48), is extended and defined by the expression:

$$\left({}_{E_\nu}\mathrm{I}_a^{(\nu)}\Omega\right)(\tau)=\int_a^{\tau}\left(\delta\,(\tau-t)-\frac{(\tau-t)^{\nu-1}}{\Gamma\,(\nu)}\right)\Omega\,(t)\,dt\ \ (\tau>a)\,, \tag{70}$$

which reduces, by using $a=0$, to an equation of the form

$$\left({}_{E_\nu}\mathrm{I}_0^{(\nu)}\Omega\right)(\tau)=\Omega\,(\tau)-\frac{1}{\Gamma\,(\nu)}\int_0^{\tau}\frac{\Omega\,(t)}{(\tau-t)^{1-\nu}}dt\ \ (\tau>0)\,, \tag{71}$$

since

$$\mathrm{L}\,[\Omega\,(\tau)]=\mathrm{L}\left[\int_0^{\tau}\delta\,(\tau-t)\,\Omega\,(t)\,dt\right], \tag{72}$$

where $\delta\,(\cdot)$ represents the Dirac delta function [3].

Thus, we have two cases:

(R1) Let $\Omega^{(1)}\in L_1^{loc}\left(\mathbb{R}_0^+\right)$. Then

$$\left({}^{C}_{E_\nu}\mathrm{D}_0^{(\nu)}\,{}_{E_\nu}\mathrm{I}_0^{(\nu)}\Omega\right)(\tau)=\Omega\,(\tau)\,, \tag{73}$$

holds true, where

$$\left({}^{C}_{E_\nu}\mathrm{D}_0^{(\nu)}\Omega\right)(\tau)=\int_0^{\tau}E_\nu\left((\tau-t)^{\nu}\right)\Omega^{(1)}\,(t)\,dt\ \ (\tau>0)\,. \tag{74}$$

(R2) Let $\Omega\in L_1\left(\mathbb{R}_0^+\right)$. Then, we have

$$\left({}^{RL}_{E_\nu}\mathrm{D}_0^{(\nu)}\,{}_{E_\nu}\mathrm{I}_0^{(\nu)}\Omega\right)(\tau)=\Omega\,(\tau)\,, \tag{75}$$

where

$$\left({}^{RL}_{E_\nu}\mathrm{D}_0^{(\nu)}\Omega\right)(\tau)=\frac{d}{d\tau}\int_0^\tau E_\nu\left((\tau-t)^\nu\right)\Omega(t)\,dt\;\;(\tau>0). \tag{76}$$

Note: Compared with the results in [24], we do not need any normalized process in Eqs. (47) and (48), and we directly select the MLF as the kernel function. Moreover, it is convenient to model complex phenomena in sciences and engineering practices with the use of GFDs in the kernel of the parameter without MLF. The Eqs. (49) and (71) are different from those derived in [24, 26]. FD of variable-order version of Eq. (47) in kernel of the negative-parametric MLF was proposed in [26].

As a direct result mentioned above, we have the following.

Let $\tau\in\mathbb{R}_0^+$, $\nu\in(0,1)$, $\Xi\in L\left(\mathbb{R}_0^+\right)$, and $\Omega^{(1)}\in L_1^{loc}\left(\mathbb{R}_0^+\right)$. The Abel type integral equation of the second kind [38]

$$\Xi(\tau)-\frac{1}{\Gamma(\nu)}\int_0^\tau\frac{\Xi(t)}{(\tau-t)^{1-\nu}}dt=\Omega(\tau), \tag{77}$$

has a unique solution

$$\Xi(t)=\int_0^\tau E_\nu\left((\tau-t)^\nu\right)\Omega^{(1)}(t)\,dt-E_\nu(\tau^\nu)\,\Omega(0), \tag{78}$$

where $\Omega(\tau=0)=\Omega(0)$.

Proof. Taking the Laplace transforms of Eqs. (77) and (78), we have from Eq. (72) that

$$\mathrm{L}[\Omega(\tau)]=\mathrm{L}[\Xi(\tau)]-\mathrm{L}\left[\frac{1}{\Gamma(\nu)}\int_0^\tau\frac{\Xi(t)}{(\tau-t)^{1-\nu}}dt\right]=\mathrm{L}[\Xi(\tau)]\left(1-\frac{1}{s^\nu}\right) \tag{79}$$

and

$$\mathrm{L}[\Xi(t)]=\frac{s^\nu}{s^\nu-1}\mathrm{L}[\Omega(\tau)]. \tag{80}$$

In fact, after rewriting Eq. (80), we get Eq. (79).

This completes the proof.

Mainardi considered Eq. (78) as [38]

$$\Xi(t)=\int_0^\tau E_\nu(t^\nu)\,\Omega^{(1)}(\tau-t)\,dt-E_\nu(\tau^\nu)\,\Omega(0). \tag{81}$$

Hille and Tamarkin presented Eq. (78) as follows (see [10] and references therein):

$$\Xi(t)=\frac{d}{d\tau}\int_0^\tau E_\nu\left((\tau-t)^\nu\right)\Omega(t)\,dt. \tag{82}$$

Remark 3. For details of the version of Eq. (57) with the parameter in kernel of the stretched exponential function, the readers should refer to [25, 26]. The version of Eq. (57) with the parameter in kernel of the stretched exponential function of variable order was reported in [26]. The revised version of Eq. (58) was also derived in [26].

Remark 4. The GFDs of the Liouville-Caputo and Riemann-Liouville types of higher order in the kernel of the MLF (1) are given as:

$$\left({}^{C}_{E_\nu}\mathrm{D}_0^{(\nu+n)}\Omega\right)(\tau)=\int_0^\tau E_\nu\left((\tau-t)^\nu\right)\Omega^{(n+1)}(t)\,dt\ (\tau>0), \tag{83}$$

$$\left({}^{RL}_{E_\nu}\mathrm{D}_0^{(\nu+n)}\Omega\right)(\tau)=\frac{d^{n+1}}{d\tau^{n+1}}\int_0^\tau E_\nu\left((\tau-t)^\nu\right)\Omega(t)\,dt\ (\tau>0), \tag{84}$$

respectively, where $\nu\in(0,1)$ and $n\in\mathbb{N}_0$.

Remark 5. For the details of fractional calculus involving extended MLFs, the readers can refer to [1–3, 24, 31–38].

Remark 6. For the kernel $o(\tau)=\tau^{1-\nu}/\Gamma(1-\nu)$, we can follow FDs of the Liouville-Caputo and Riemann-Liouville types, given by [1, 18, 19]:

$$\left({}^{C}\mathrm{D}_0^{(\nu)}\Omega\right)(\tau)=\frac{1}{\Gamma(1-\nu)}\int_0^\tau \frac{\Omega^{(1)}(t)}{(\tau-t)^{1-\nu}}dt\ (\tau>0), \tag{85}$$

$$\left({}^{RL}\mathrm{D}_0^{(\nu)}\Omega\right)(\tau)=\frac{1}{\Gamma(1-\nu)}\frac{d}{d\tau}\int_0^\tau \frac{\Omega(t)}{(\tau-t)^{1-\nu}}dt\ (\tau>0), \tag{86}$$

respectively, where [1]

$$\left({}^{C}\mathrm{D}_{+0}^{(\nu)}\Omega\right)(\tau)=\left({}^{RL}\mathrm{D}_{+0}^{(\nu)}\Omega\right)(\tau)-\frac{\tau^{1-\nu}\Omega(a)}{\Gamma(1-\nu)}. \tag{87}$$

In fact, we can extend the GFDs of the Liouville-Caputo and Riemann-Liouville types in the kernel of the MLF (1), defined as:

$$\left({}^{C}_{E_\nu(-)}\mathrm{D}_0^{(\nu)}\Omega\right)(\tau)=\int_0^\tau E_\nu\left(-(\tau-t)^\nu\right)\Omega^{(1)}(t)\,dt\ (\tau>0), \tag{88}$$

$$\left({}^{RL}_{E_\nu(-)}\mathrm{D}_0^{(\nu)}\Omega\right)(\tau)=\frac{d}{d\tau}\int_0^\tau E_\nu\left(-(\tau-t)^\nu\right)\Omega(t)\,dt\ (\tau>0), \tag{89}$$

respectively, where

$$\left({}^{C}_{E_\nu(-)}\mathrm{D}_0^{(\nu)}\Omega\right)(\tau)=\left({}^{RL}_{E_\nu(-)}\mathrm{D}_0^{(\nu)}\Omega\right)(\tau)-E_\nu\left(-\tau^\nu\right)\Omega(0). \tag{90}$$

Similarly, the corresponding general fractional integral is defined as:

$$\left({}_{E_\nu(-)}\mathrm{I}_0^{(\nu)}\Omega\right)(\tau)=\Omega(\tau)+\frac{1}{\Gamma(\nu)}\int_0^\tau \frac{\Omega(t)}{(\tau-t)^{1-\nu}}dt\ (\tau>0). \tag{91}$$

Let $\tau\in\mathbb{R}_0^+$, $\nu\in(0,1)$, $\Xi\in L\left(\mathbb{R}_0^+\right)$, and $\Omega^{(1)}\in L_1^{loc}\left(\mathbb{R}_0^+\right)$. The Abel type integral equation of the second kind [38]

$$\Xi(\tau)+\frac{1}{\Gamma(\nu)}\int_0^\tau \frac{\Xi(t)}{(\tau-t)^{1-\nu}}dt = \Omega(\tau), \tag{92}$$

has a unique solution

$$\Xi(t)=\int_0^\tau E_\nu\left(-(\tau-t)^\nu\right)\Omega^{(1)}(t)\,dt - E_\nu(-\tau^\nu)\,\Omega(0), \tag{93}$$

where $\Omega(\tau=0)=\Omega(0)$.

Proof. In a similar manner, for obtaining the Laplace transforms of Eqs. (77) and (78), we have from Eq. (72) that

$$\mathrm{L}[\Omega(\tau)] = \mathrm{L}[\Xi(\tau)] + \mathrm{L}\left[\frac{1}{\Gamma(\nu)}\int_0^\tau \frac{\Xi(t)}{(\tau-t)^{1-\nu}}dt\right] = \mathrm{L}[\Xi(\tau)]\left(1+\frac{1}{s^\nu}\right) \tag{94}$$

and

$$\mathrm{L}[\Xi(t)] = \frac{s^\nu}{s^\nu+1}\mathrm{L}[\Omega(\tau)]. \tag{95}$$

We get Eq. (95) after arranging Eq. (94).

Therefore, the proof is completed.

The Laplace transforms of the GFDs in the kernels of extended MLFs are as follows:

$$\begin{aligned}&\mathrm{L}\left[\left({}_{E_{\varphi,\phi}}^{C}\mathrm{D}_0^{(\nu)}\Omega\right)(\tau)\right] = {}_2\Psi_{n+1}\left[\begin{array}{cc} (\nu,1)(\phi,\varphi); & \\ (\nu_1,\upsilon_1)\ldots(\nu_n,\upsilon_n)(0,\varphi); & \frac{1}{s^\nu}\end{array}\right]\\ &\times\left(\Omega(s)-\frac{\Omega(0)}{s}\right),\end{aligned} \tag{96}$$

$$\begin{aligned}&\mathrm{L}\left[\left({}_{E_{\varphi,\phi}(-)}^{C}\mathrm{D}_0^{(\nu)}\Omega\right)(\tau)\right] = {}_2\Psi_{n+1}\left[\begin{array}{cc} (\nu,1)(\phi,\varphi); & \\ (\nu_1,\upsilon_1)\ldots(\nu_n,\upsilon_n)(0,\varphi); & -\frac{1}{s^\nu}\end{array}\right]\\ &\times\left(\Omega(s)-\frac{\Omega(0)}{s}\right),\end{aligned} \tag{97}$$

$$\mathrm{L}\left[\left({}_{E_{\nu,\upsilon}^{\varphi,\phi}}^{C}\mathrm{D}_0^{(\nu)}\Omega\right)(\tau)\right] = {}_2\Psi_2\left[\begin{array}{cc} (\nu,1)(\phi,\varphi); & \\ (\nu,\upsilon)(0,\varphi); & \frac{1}{s^\nu}\end{array}\right]\left(\Omega(s)-\frac{\Omega(0)}{s}\right), \tag{98}$$

$$\mathrm{L}\left[\left({}_{E_{\nu,\upsilon}^{\varphi,\phi}(-)}^{C}\mathrm{D}_0^{(\nu)}\Omega\right)(\tau)\right] = {}_2\Psi_2\left[\begin{array}{cc} (\nu,1)(\phi,\varphi); & \\ (\nu,\upsilon)(0,\varphi); & -\frac{1}{s^\nu}\end{array}\right]\left(\Omega(s)-\frac{\Omega(0)}{s}\right), \tag{99}$$

$$\mathrm{L}\left[\left({}_{E_{\nu,\upsilon}^{\varphi}}^{C}\mathrm{D}_0^{(\nu)}\Omega\right)(\tau)\right] = {}_2\Psi_2\left[\begin{array}{cc} (\nu,1)(1,\varphi); & \\ (\nu,\upsilon)(0,\varphi); & \frac{1}{s^\nu}\end{array}\right]\left(\Omega(s)-\frac{\Omega(0)}{s}\right), \tag{100}$$

$$\mathrm{L}\left[\left({}_{E^{\varphi}_{\nu,\upsilon}(-)}^{C}\mathrm{D}_0^{(\nu)}\Omega\right)(\tau)\right] = {}_2\Psi_2\left[\begin{array}{ll}(\nu,1)(1,\varphi); & -\frac{1}{s^{\nu}}\\ (\nu,\upsilon)(0,\varphi); & \end{array}\right]\left(\Omega(s)-\frac{\Omega(0)}{s}\right), \tag{101}$$

$$\mathrm{L}\left[\left({}_{E_{\nu,\upsilon}}^{C}\mathrm{D}_0^{(\nu)}\Omega\right)(\tau)\right] = {}_2\Psi_2\left[\begin{array}{ll}(\nu,1)(1,1); & \frac{1}{s^{\nu}}\\ (\nu,\upsilon)(0,1); & \end{array}\right]\left(\Omega(s)-\frac{\Omega(0)}{s}\right), \tag{102}$$

$$\mathrm{L}\left[\left({}_{E_{\nu,\upsilon}(-)}^{C}\mathrm{D}_0^{(\nu)}\Omega\right)(\tau)\right] = {}_2\Psi_2\left[\begin{array}{ll}(\nu,1)(1,1); & -\frac{1}{s^{\nu}}\\ (\nu,\upsilon)(0,1); & \end{array}\right]\left(\Omega(s)-\frac{\Omega(0)}{s}\right), \tag{103}$$

$$\mathrm{L}\left[\left({}_{E_{\nu}}^{C}\mathrm{D}_0^{(\nu)}\Omega\right)(\tau)\right] = {}_2\Psi_2\left[\begin{array}{ll}(\nu,1)(1,1); & \frac{1}{s^{\nu}}\\ (\nu,1)(0,1); & \end{array}\right]\left(\Omega(s)-\frac{\Omega(0)}{s}\right), \tag{104}$$

$$\mathrm{L}\left[\left({}_{E_{\nu}(-)}^{C}\mathrm{D}_0^{(\nu)}\Omega\right)(\tau)\right] = {}_2\Psi_2\left[\begin{array}{ll}(\nu,1)(1,1); & -\frac{1}{s^{\nu}}\\ (\nu,1)(0,1); & \end{array}\right]\left(\Omega(s)-\frac{\Omega(0)}{s}\right), \tag{105}$$

$$\mathrm{L}\left[\left({}_{E_{\varphi,\phi}}^{RL}\mathrm{D}_0^{(\nu)}\Omega\right)(\tau)\right] = {}_2\Psi_{n+1}\left[\begin{array}{ll}(\nu,1)(\phi,\varphi); & \frac{1}{s^{\nu}}\\ (\nu_1,\upsilon_1)\ldots(\nu_n,\upsilon_n)(0,\varphi); & \end{array}\right]\Omega(s), \tag{106}$$

$$\mathrm{L}\left[\left({}_{E_{\varphi,\phi}(-)}^{RL}\mathrm{D}_0^{(\nu)}\Omega\right)(\tau)\right] = {}_2\Psi_{n+1}\left[\begin{array}{ll}(\nu,1)(\phi,\varphi); & -\frac{1}{s^{\nu}}\\ (\nu_1,\upsilon_1)\ldots(\nu_n,\upsilon_n)(0,\varphi); & \end{array}\right]\Omega(s), \tag{107}$$

$$\mathrm{L}\left[\left({}_{E^{\varphi,\phi}_{\nu,\upsilon}}^{RL}\mathrm{D}_0^{(\nu)}\Omega\right)(\tau)\right] = {}_2\Psi_2\left[\begin{array}{ll}(\nu,1)(\phi,\varphi); & \frac{1}{s^{\nu}}\\ (\nu,\upsilon)(0,\varphi); & \end{array}\right]\Omega(s), \tag{108}$$

$$\mathrm{L}\left[\left({}_{E^{\varphi,\phi}_{\nu,\upsilon}(-)}^{RL}\mathrm{D}_0^{(\nu)}\Omega\right)(\tau)\right] = {}_2\Psi_2\left[\begin{array}{ll}(\nu,1)(\phi,\varphi); & -\frac{1}{s^{\nu}}\\ (\nu,\upsilon)(0,\varphi); & \end{array}\right]\Omega(s), \tag{109}$$

$$\mathrm{L}\left[\left({}_{E^{\varphi}_{\nu,\upsilon}}^{C}\mathrm{D}_0^{(\nu)}\Omega\right)(\tau)\right] = {}_2\Psi_2\left[\begin{array}{ll}(\nu,1)(1,\varphi); & \frac{1}{s^{\nu}}\\ (\nu,\upsilon)(0,\varphi); & \end{array}\right]\Omega(s), \tag{110}$$

$$\mathrm{L}\left[\left({}_{E^{\varphi}_{\nu,\upsilon}(-)}^{C}\mathrm{D}_0^{(\nu)}\Omega\right)(\tau)\right] = {}_2\Psi_2\left[\begin{array}{ll}(\nu,1)(1,\varphi); & -\frac{1}{s^{\nu}}\\ (\nu,\upsilon)(0,\varphi); & \end{array}\right]\Omega(s), \tag{111}$$

$$\mathrm{L}\left[\left({}_{E_{\nu,\upsilon}}^{RL}\mathrm{D}_0^{(\nu)}\Omega\right)(\tau)\right] = {}_2\Psi_2\left[\begin{array}{ll}(\nu,1)(1,1); & \frac{1}{s^{\nu}}\\ (\nu,\upsilon)(0,1); & \end{array}\right]\Omega(s), \tag{112}$$

$$\mathrm{L}\left[\left({}_{E_{\nu,\upsilon}(-)}^{RL}\mathrm{D}_0^{(\nu)}\Omega\right)(\tau)\right] = {}_2\Psi_2\left[\begin{array}{ll}(\nu,1)(1,1); & -\frac{1}{s^{\nu}}\\ (\nu,\upsilon)(0,1); & \end{array}\right]\Omega(s), \tag{113}$$

$$\mathrm{L}\left[\left({}_{E_\nu}^{RL}\mathrm{D}_0^{(\nu)}\Omega\right)(\tau)\right] = {}_2\Psi_2\left[\begin{array}{cc} (\nu,1)(1,1); & \frac{1}{s^\nu} \\ (\nu,1)(0,1); & \end{array}\right]\Omega(s), \tag{114}$$

$$\mathrm{L}\left[\left({}_{E_\nu(-)}^{RL}\mathrm{D}_0^{(\nu)}\Omega\right)(\tau)\right] = {}_2\Psi_2\left[\begin{array}{cc} (\nu,1)(1,1); & -\frac{1}{s^\nu} \\ (\nu,1)(0,1); & \end{array}\right]\Omega(s). \tag{115}$$

3. ON ANOMALOUS DIFFUSION MODELS OF FRACTIONAL ORDER

In this section, we consider two examples for illustrating the general Liouville-Caputo and Riemann-Liouville types of fractional-time derivative anomalous diffusion equations.

Example 1

The general Liouville-Caputo type fractional-time derivative anomalous diffusion equation within the Saxena-Nishimoto-type MLF can be written as:

$$\frac{\partial^\nu \mathrm{X}_\nu(\eta,\tau)}{\partial\tau^\nu} = \kappa\frac{\partial^2 \mathrm{X}_\nu(\eta,\tau)}{\partial\eta^2},\ \eta>0,\ \tau>0, \tag{116}$$

with the initial and boundary conditions:

$$\mathrm{X}_\nu(\eta,0)=0,\ \eta>0, \tag{117}$$

$$\mathrm{X}_\nu(0,\tau)=\delta(\tau),\ \tau>0, \tag{118}$$

$$\mathrm{X}_\nu(\eta,0)\to 0,\ as\ \eta\to\infty,\ \tau>0, \tag{119}$$

where κ is the thermal diffusivity, $\delta(\tau)$ is the Dirac function [3], and the general Liouville-Caputo fractional-time derivative is defined as:

$$\frac{\partial^\nu \mathrm{X}_\nu(\eta,\tau)}{\partial\tau^\nu} = \int_0^\tau E_{\varphi,\phi}\left((\nu_1,\upsilon_1),...,(\nu_n,\upsilon_n);-(\tau-t)^\nu\right)\frac{\partial}{\partial t}\mathrm{X}_\nu(\eta,t)\,dt. \tag{120}$$

Applying the Laplace transform of Eq. (116), it follows that

$${}_2\Psi_{n+1}\left[\begin{array}{cc} (\nu,1)(\phi,\varphi); & -\frac{1}{s^\nu} \\ (\nu_1,\upsilon_1)...(\nu_n,\upsilon_n)(0,\varphi); & \end{array}\right]\mathrm{X}_\nu(\eta,s) = \kappa\frac{\partial^2 \mathrm{X}_\nu(\eta,s)}{\partial\eta^2}, \tag{121}$$

where

$$\mathrm{X}_\nu(0,s)=1. \tag{122}$$

Thus, we obtain

$$\mathrm{X}_{\nu}\left(\eta,s\right)=\exp\left(-\eta\left(\frac{{}_{2}\Psi_{n+1}\left[\begin{array}{ll}(\nu,1)\,(\phi,\varphi)\,; & \\ (\nu_{1},\upsilon_{1})\ldots(\nu_{n},\upsilon_{n})\,(0,\varphi)\,; & -\frac{1}{s^{\nu}}\end{array}\right]}{\kappa}\right)^{\frac{1}{2}}\right). \tag{123}$$

We can rewrite Eq. (123) as the series:

$$\mathrm{X}_{\nu}\left(\eta,s\right)=\sum_{i=0}^{\infty}\mathrm{X}_{\nu,i}\left(\eta,s\right)=\sum_{i=0}^{\infty}\frac{\eta^{i}\left(\frac{{}_{2}\Psi_{n+1}\left[\begin{array}{ll}(\nu,1)\,(\phi,\varphi)\,; & \\ (\nu_{1},\upsilon_{1})\ldots(\nu_{n},\upsilon_{n})\,(0,\varphi)\,; & -\frac{1}{s^{\nu}}\end{array}\right]}{\kappa}\right)^{\frac{i}{2}}}{\Gamma\left(i+1\right)}, \tag{124}$$

where $i\in\mathbb{N}_{0}$.

From Eq. (124) we get the equations of the series form:

$$\mathrm{X}_{\nu,0}\left(\eta,s\right)=1, \tag{125}$$

$$\mathrm{X}_{\nu,1}\left(\eta,s\right)=-\eta\left(\frac{{}_{2}\Psi_{n+1}\left[\begin{array}{ll}(\nu,1)\,(\phi,\varphi)\,; & \\ (\nu_{1},\upsilon_{1})\ldots(\nu_{n},\upsilon_{n})\,(0,\varphi)\,; & -\frac{1}{s^{\nu}}\end{array}\right]}{\kappa}\right)^{\frac{1}{2}}, \tag{126}$$

$$\vdots$$

$$\mathrm{X}_{\nu,i}\left(\eta,s\right)=\frac{(-\eta)^{i}}{\Gamma\left(1+i\right)}\left(\frac{{}_{2}\Psi_{n+1}\left[\begin{array}{ll}(\nu,1)\,(\phi,\varphi)\,; & \\ (\nu_{1},\upsilon_{1})\ldots(\nu_{n},\upsilon_{n})\,(0,\varphi)\,; & -\frac{1}{s^{\nu}}\end{array}\right]}{\kappa}\right)^{\frac{i}{2}} \tag{127}$$

and so on.

Therefore, we obtain the following solutions in the series form:

$$\mathrm{X}_{\nu,0}\left(\eta,\tau\right)=1, \tag{128}$$

$$\mathrm{X}_{\nu,1}\left(\eta,\tau\right)=-\eta\Theta_{\frac{1}{2}}\left(\tau\right), \tag{129}$$

$$\vdots$$

$$\mathrm{X}_{\nu,i}(\eta,\tau) = \frac{(-\eta)^i}{\Gamma(1+i)}\Theta_{\frac{i}{2}}(\tau), \tag{130}$$

where

$$\mathrm{L}\left[\Theta_{\frac{i}{2}}(\tau)\right] = \left(\frac{{}_2\Psi_{n+1}\left[\begin{array}{ll} (\nu,1)(\phi,\varphi); & \\ (\nu_1,\upsilon_1)\ldots(\nu_n,\upsilon_n)(0,\varphi); & -\frac{1}{s^\nu} \end{array} \right]}{\kappa} \right)^{\frac{i}{2}}. \tag{131}$$

Example 2

The general Riemann-Liouville type fractional-time derivative anomalous diffusion equation within the Saxena-Nishimoto-type MLF can be presented as:

$$\frac{\partial^\nu \mathrm{X}_\nu(\eta,\tau)}{\partial\tau^\nu} = \kappa\frac{\partial^2 \mathrm{X}_\nu(\eta,\tau)}{\partial\eta^2},\ \eta>0,\ \tau>0, \tag{132}$$

with the initial and boundary conditions:

$$\left[\int_0^\tau E_{\varphi,\phi}((\nu_1,\upsilon_1),\ldots,(\nu_n,\upsilon_n);(\tau-t)^\nu)\,\mathrm{X}_\nu(\eta,t)\,dt\right]|_{\tau=0} = 0,\ \eta>0, \tag{133}$$

$$\mathrm{X}_\nu(0,\tau) = \pi(\tau),\ \tau>0, \tag{134}$$

$$\mathrm{X}_\nu(\eta,0)\to 0, for\ \eta\to\infty,\ \tau>0, \tag{135}$$

where κ is the thermal diffusivity, and the general Riemann-Liouville type fractional-time derivative operator is defined as:

$$\frac{\partial^\nu \mathrm{X}_\nu(\eta,\tau)}{\partial\tau^\nu} = \frac{\partial}{\partial t}\int_0^\tau E_{\varphi,\phi}((\nu_1,\upsilon_1),\ldots,(\nu_n,\upsilon_n);-(\tau-t)^\nu)\,\mathrm{X}_\nu(\eta,t)\,dt. \tag{136}$$

Taking the Laplace transform of Eq. (132), we obtain the following equations of the form:

$${}_2\Psi_{n+1}\left[\begin{array}{ll} (\nu,1)(\phi,\varphi); & \\ (\nu_1,\upsilon_1)\ldots(\nu_n,\upsilon_n)(0,\varphi); & -\frac{1}{s^\nu} \end{array} \right]\mathrm{X}_\nu(\eta,s) = \kappa\frac{\partial^2 \mathrm{X}_\nu(\eta,s)}{\partial\eta^2}, \tag{137}$$

which leads to

$$\frac{{}_2\Psi_{n+1}\left[\begin{array}{ll} (\nu,1)(\phi,\varphi); & \\ (\nu_1,\upsilon_1)\ldots(\nu_n,\upsilon_n)(0,\varphi); & -\frac{1}{s^\nu} \end{array} \right]}{\kappa}\mathrm{X}_\nu(\eta,s) = \frac{\partial^2 \mathrm{X}_\nu(\eta,s)}{\partial\eta^2}. \tag{138}$$

Due to Eq. (134), we have

$$\mathrm{X}_\nu(0,s) = \pi(s) \tag{139}$$

such that

$$\mathrm{X}_{\nu}(\eta,s)=\pi(s)\exp\left(-\eta\left(\frac{{}_{2}\Psi_{n+1}\left[\begin{array}{ll}(\nu,1)(\phi,\varphi); & \\ (\nu_1,\upsilon_1)\ldots(\nu_n,\upsilon_n)(0,\varphi); & -\frac{1}{s^{\nu}}\end{array}\right]}{\kappa}\right)^{\frac{1}{2}}\right). \tag{140}$$

In a similar way, we rewrite Eq. (140) as the series solution:

$$\mathrm{X}_{\nu}(\eta,s)=\sum_{i=0}^{\infty}\mathrm{X}_{\nu,i}(\eta,s), \tag{141}$$

where

$$\sum_{i=0}^{\infty}\mathrm{X}_{\nu,i}(\eta,s)=\pi(s)\sum_{i=0}^{\infty}\frac{\eta^{i}\left(\frac{{}_{2}\Psi_{n+1}\left[\begin{array}{ll}(\nu,1)(\phi,\varphi); & \\ (\nu_1,\upsilon_1)\ldots(\nu_n,\upsilon_n)(0,\varphi); & -\frac{1}{s^{\nu}}\end{array}\right]}{\kappa}\right)^{\frac{i}{2}}}{\Gamma(i+1)}.$$

Thus, from Eq. (141) we get the expressions of the series form:

$$\mathrm{X}_{\nu,0}(\eta,s)=\pi(s), \tag{142}$$

$$\mathrm{X}_{\nu,1}(\eta,s)=-\pi(s)\,\eta\left(\frac{{}_{2}\Psi_{n+1}\left[\begin{array}{ll}(\nu,1)(\phi,\varphi); & \\ (\nu_1,\upsilon_1)\ldots(\nu_n,\upsilon_n)(0,\varphi); & -\frac{1}{s^{\nu}}\end{array}\right]}{\kappa}\right)^{\frac{1}{2}}, \tag{143}$$

$$\vdots$$

$$\mathrm{X}_{\nu,i}(\eta,s)=\pi(s)\frac{\eta^{3}}{\Gamma(1+i)}\left(\frac{{}_{2}\Psi_{n+1}\left[\begin{array}{ll}(\nu,1)(\phi,\varphi); & \\ (\nu_1,\upsilon_1)\ldots(\nu_n,\upsilon_n)(0,\varphi); & -\frac{1}{s^{\nu}}\end{array}\right]}{\kappa}\right)^{\frac{i}{2}} \tag{144}$$

and so on.

Thus, we have the following solutions in the series form

$$\mathrm{X}_{\nu,0}(\eta,\tau)=\pi(\tau), \tag{145}$$

$$X_{\nu,1}(\eta,\tau) = -\eta \int_0^\tau \pi(\tau - t)\,\Theta_{\frac{1}{2}}(t)\,dt, \tag{146}$$

$$\vdots$$

$$X_{\nu,i}(\eta,\tau) = \frac{(-\eta)^i}{\Gamma(1+i)} \int_0^\tau \pi(\tau - t)\,\Theta_{\frac{i}{2}}(t)\,dt, \tag{147}$$

such that

$$X_{\nu}(\eta,\tau) = \pi(\tau) + \sum_{i=1}^{\infty} \frac{(-\eta)^i}{\Gamma(1+i)} \int_0^\tau \pi(\tau - t)\,\Theta_{\frac{i}{2}}(t)\,dt. \tag{148}$$

Since the kernel functions are generalized as the other MLFs, the results are important for discussing the anomalous diffusion models within GFDs in kernels of extended MLFs.

4. CONCLUSION

In this work, we formulated the GFDs within the kernels of the extended MLFs from the GFC point of view. The Laplace transforms of the GFDs were proposed and the anomalous diffusion models were considered. The reported results represent new mathematical tools to describe the power-law behaviors of a series of complex physical phenomena.

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