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NEW GENERAL FRACTIONAL-ORDER RHEOLOGICAL MODELS WITH KERNELS OF MITTAG-LEFFLER FUNCTIONS

XIAO-JUN YANG[1,2]

[1] State Key Laboratory for Geomechanics and Deep Underground Engineering, China University of Mining and Technology,
Xuzhou 221116, Peoples Republic of China
Email: *dyangxiaojun@163.com*
[2] School of Mechanics and Civil Engineering, China University of Mining and Technology,
Xuzhou 221116, Peoples Republic of China

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Abstract. In this paper, we consider new fractional-order Maxwell and Voigt models within the framework of the general fractional derivatives (GFDs). The operators are considered in the sense of Liouville-Caputo and Riemann-Liouville types GFDs involving the kernels of the Mittag-Leffler functions. The creep and relaxation characteristics for the fractional-order models are also discussed in detail. The formulations are proposed as useful tools to describe the complex behaviors of the general fractional-order viscoelasticity with memory effect.

Key words: General fractional derivatives, Mittag-Leffler function, fractional-order Voigt element, fractional-order Maxwell element, memory effect.

1. INTRODUCTION

The fractional derivatives (FDs) of the Liouville-Caputo and Riemann-Liouville types [1–9] have been recently used by scientists and engineers to describe several complex phenomena in mathematical physics and engineering [10–14]. The general fractional calculus (GFC) has been used to model many real-world problems in mathematical physics [15–17]. The general fractional derivatives (GFDs) within the kernels of the Mittag-Leffler functions (MLFs) were proposed to model the anomalous diffusion problems (see [18]).

Following the history of the fractional-order linear viscoelasticity [2, 19–28], there exist two viewpoints: the power-law stress relaxation [20, 21, 26] and the generalized MLF [22, 23]. The general laws of deformation were considered in [19–21, 24–29]. However, the general laws of deformation involving the positive- and negative-parametric MLFs have not been yet reported, to the best of our knowledge.

Motivated by the ideas mentioned above, the main aim of the paper is to consider the approximations of the power-law stress relaxation behaviors, and to develop new rheological models described by the GFC within the MLF kernels.

The structure of the paper is as follows. In Section 2, we introduce the GFDs in the kernels of the MLFs. In Section 3, we propose the rheological models involving the GFDs within the kernels of the MLFs. Finally, the conclusion is given in Section 4.

2. PRELIMINARIES ON THE GFDS IN THE KERNELS OF THE MLFS

In this Section, we briefly present the GFC in the MLF kernels used in this paper.

2.1. THE MLFS

Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}_0^+$, $\mathbb{N}$, and $\mathbb{N}_0$ be the sets of complex numbers, real numbers , non-negative real numbers, positive integers, and let $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.

The MLF, introduced by the Swedish mathematician Gosta Mittag-Leffler in the year 1903, is defined by [30]:

$$E_\alpha(h) = \sum_{\eta=0}^{\infty} \frac{h^\eta}{\Gamma(\eta\alpha+1)}, \tag{1}$$

where $h, \alpha \in \mathbb{C}$, $\Re(\alpha) \in \mathbb{R}_0^+$, $\eta \in \mathbb{N}$, and $\Gamma(\cdot)$ is the familiar Gamma function [1].

The Laplace transform of the MLF is as follows [27]:

$$\mathrm{S}[E_\alpha(\nu h^\alpha)] = \frac{1}{s} {}_2\Psi_2 \left[\begin{array}{cc} (\alpha,1)(1,1); & \frac{\nu}{s^\alpha} \\ (\alpha,1)(0,1); & \end{array} \right] = \frac{s^{\alpha-1}}{s^\alpha - \nu}, \tag{2}$$

where ν is a constant, the generalized Wright function ${}_\varpi\Psi_\omega\,(\varpi, \omega \in \mathbb{N}_0)$ is given as [31]:

$$\begin{array}{l} {}_\varpi\Psi_\omega \left[\begin{array}{cc} (b_1,B_1),...,(b_\varpi,B_\varpi); & \eta \\ (c_1,C_1),...,(c_\omega,C_\omega); & \end{array} \right] = \sum\limits_{\eta=0}^{\infty} \frac{\prod_{j=0}^{\varpi}\Gamma(\kappa b_j+B_j)\cdots\Gamma(\kappa b_j+B_j)}{\prod_{j=0}^{\omega}\Gamma(\kappa c_j+C_j)\cdots\Gamma(\kappa c_j+C_j)} \\ \times \frac{h^\eta}{\Gamma(\eta+1)}, \end{array} \tag{3}$$

$\left(\sum\limits_{j=1}^{m} \Re(B_j) > 0;\ \sum\limits_{j=1}^{m} \Re(C_j) > 0; 1 + \left(\sum\limits_{j=1}^{m} \Re(C_j) - \sum\limits_{j=1}^{m} \Re(B_j)\right) \geq 0\right)$ and the Laplace transform is defined as [27]:

$$\mathrm{S}[\mathrm{F}(h)] = \mathrm{F}(s) := \int_0^\infty e^{-sh}\mathrm{F}(h)\,dh. \tag{4}$$

As first generation of the MLF, the two-paramedic MLF, proposed by Wiman in the year 1905, is defined as [32]:

$$E_{\alpha,\upsilon}(h)=\sum_{\eta=0}^{\infty}\frac{h^{\eta}}{\Gamma(\eta\alpha+\upsilon)},\tag{5}$$

where $h,\alpha,\upsilon\in\mathbb{C}$, $\Re(\alpha),\Re(\upsilon)\in\mathbb{R}_0^+$, and $\eta\in\mathbb{N}$.

The Laplace transform of the two-paramedic MLF is as follows:

$$\mathrm{S}\left[E_{\alpha,\alpha+1}(\nu h^{\alpha})\right]=\frac{1}{s(s^{\alpha}-\nu)},\tag{6}$$

which is derived from the equation [33]:

$$\mathrm{S}\left[h^{\upsilon-1}E_{\alpha,\upsilon}(\nu h^{\alpha})\right]=\frac{s^{\alpha-\upsilon}}{s^{\alpha}-\nu}.\tag{7}$$

2.2. GFDS IN THE KERNELS OF THE MITTAG-LEFFLER FUNCTIONS

The Liouville-Caputo and Riemann-Liouville GFDs in the kernel of the positive-parametric MLF are defined as [18]:

$$\left({}^{C}_{E_{\alpha}(+)}\mathrm{D}_0^{(\alpha)}\mathrm{F}\right)(h)=\int_0^h E_{\alpha}\left((h-t)^{\alpha}\right)\mathrm{F}^{(1)}(t)\,dt\ \ (h>0),\tag{8}$$

$$\left({}^{RL}_{E_{\alpha}(+)}\mathrm{D}_0^{(\alpha)}\mathrm{F}\right)(h)=\frac{d}{dh}\int_0^h E_{\alpha}\left((h-t)^{\alpha}\right)\mathrm{F}(t)\,dt\ \ (h>0),\tag{9}$$

respectively, where

$$\left({}^{C}_{E_{\alpha}(+)}\mathrm{D}_0^{(\alpha)}\mathrm{F}\right)(h)=\left({}^{RL}_{E_{\alpha}(+)}\mathrm{D}_0^{(\alpha)}\mathrm{F}\right)(h)-E_{\alpha}(h^{\alpha})\mathrm{F}(0)\ \ (h>0).$$

The Liouville-Caputo and Riemann-Liouville GFDs in the kernel of the negative-parametric MLF are defined as [18]:

$$\left({}^{C}_{E_{\alpha}(-)}\mathrm{D}_0^{(\alpha)}\mathrm{F}\right)(h)=\int_0^h E_{\alpha}\left(-(h-t)^{\alpha}\right)\mathrm{F}^{(1)}(t)\,dt\ \ (h>0),\tag{10}$$

$$\left({}^{RL}_{E_{\alpha}(-)}\mathrm{D}_0^{(\alpha)}\mathrm{F}\right)(h)=\frac{d}{dh}\int_0^h E_{\alpha}\left(-(h-t)^{\alpha}\right)\mathrm{F}(t)\,dt\ \ (h>0),\tag{11}$$

respectively, where

$$\left({}^{C}_{E_{\alpha}(-)}\mathrm{D}_0^{(\alpha)}\mathrm{F}\right)(h)=\left({}^{RL}_{E_{\alpha}(-)}\mathrm{D}_0^{(\alpha)}\mathrm{F}\right)(h)-E_{\alpha}(-h^{\alpha})\mathrm{F}(0)\ \ (h>0).$$

The Laplace transforms of the GFDs in the positive- and negative-parametric MLFs kernels are as follows [18]:

$$\mathrm{S}\left[\left({}^{C}_{E_{\alpha}(+)}\mathrm{D}_0^{(\alpha)}\mathrm{F}\right)(h)\right]=\frac{s^{\alpha-1}}{s^{\alpha}-1}\left(s\mathrm{F}(s)-\mathrm{F}(0)\right),\tag{12}$$

$$\mathrm{S}\left[\left({}^{RL}_{E_\alpha(+)}\mathrm{D}_0^{(\alpha)}\mathrm{F}\right)(h)\right]=\frac{s^\alpha}{s^\alpha-1}\mathrm{F}(s),\tag{13}$$

$$\mathrm{S}\left[\left({}^{C}_{E_\alpha(-)}\mathrm{D}_0^{(\alpha)}\mathrm{F}\right)(h)\right]=\frac{s^{\alpha-1}}{s^\alpha+1}\left(s\mathrm{F}(s)-\mathrm{F}(0)\right),\tag{14}$$

$$\mathrm{S}\left[\left({}^{RL}_{E_\alpha(-)}\mathrm{D}_0^{(\alpha)}\mathrm{F}\right)(h)\right]=\frac{s^\alpha}{s^\alpha+1}\mathrm{F}(s).\tag{15}$$

3. GENERAL FRACTIONAL-ORDER MODELS WITHIN THE GFDS

In order to show the characteristics of real materials using the Nutting equation given by [2, 19]:

$$\varepsilon_\alpha(\tau)=\Upsilon\tau^g\sigma_\alpha^f,\tag{16}$$

where Υ, g, and f are the model parameters, ε_α is the strain, σ_α is the stress, and τ is the time, we may write the equation of the power-law stress relaxation (for $f=1$) as [25]:

$$\varepsilon_\alpha(\tau)=\Upsilon\tau^g\sigma_\alpha.\tag{17}$$

Let us rewrite Eq. (17) as

$$\varepsilon_\alpha(\tau)=\Upsilon\tau^{-|g|}\sigma_\alpha\ \left(|g|=-g\right)$$

or

$$\varepsilon_\nu(\tau)=\Upsilon\tau^{|g|}\sigma_\nu\ \left(|g|=g\right)$$

Following the idea mentioned in [27], we have

$$E_\alpha(-\tau^\alpha)\propto\tau^{-\alpha}$$

and

$$E_\alpha(\tau^\alpha)\propto\tau^\alpha,$$

where $\tau\to\infty$, such that we have for $0<\alpha<1$

$$\sigma_\alpha=\Upsilon\tau^{-\alpha}\varepsilon_\alpha,\ \sigma_\alpha=\Upsilon\tau^{\alpha}\varepsilon_\alpha,\ \sigma_\alpha=\Upsilon\varepsilon_\alpha E_\alpha(-\tau^\alpha),\sigma_\alpha=\Upsilon\ E_\alpha(\tau^\alpha)\varepsilon_\alpha,$$

which yield the (negative and positive) power- and (negative and positive) Mittag-Leffler-law stress relaxation, and their plots are illustrated in Fig. 1.

We notice that the first term (the so-called negative-power-law stress relaxation) was adopted to describe the power-law stress within the frameworks of the FDs within the power-law kernel [2, 21, 24, 25, 27].

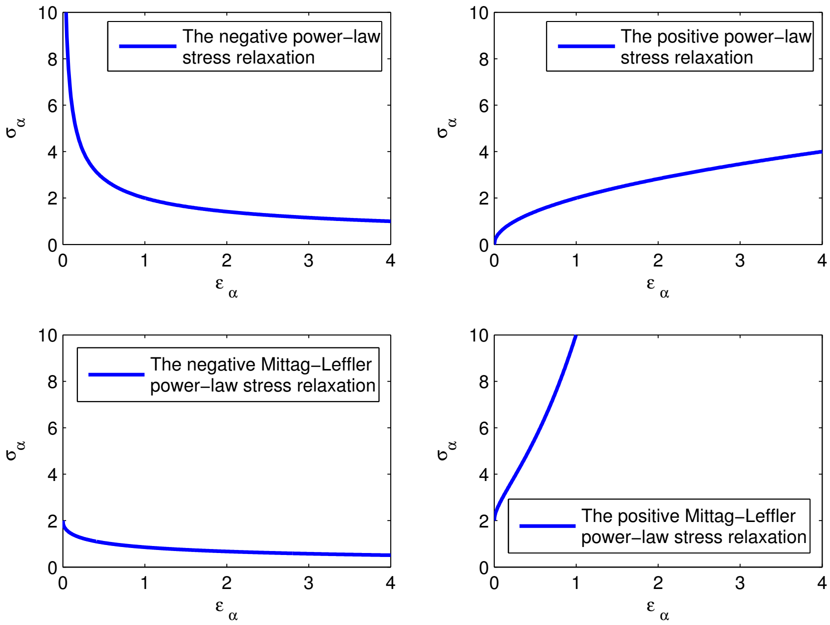

Fig. 1 – The plots of the stress relaxation for the parameter $\alpha = 0.5$: The negative power-law stress relaxation; The positive power-law stress relaxation; The negative Mittag-Leffler power-law stress relaxation; The positive Mittag-Leffler power-law stress relaxation.

3.1. FRACTIONAL-ORDER SPRING-DASHPOT ELEMENTS

The spring element (SE) follows the Hooke's law, whose the constructive equation (CE) is given as:

$$\sigma_\alpha(\tau) = \Pi \varepsilon_\alpha(\tau), \tag{18}$$

where Π is the Young's modulus of the material.

The general fractional-order dashpot elements (GFODE), adopted to describe the viscous flow, follows the general Newton's laws, whose the CEs are considered as follows. Let Υ be the viscosity. Then we have the following:

(C1) The GFODE involving the Liouville-Caputo FD in the kernel of the positive-parametric MLF is represented as:

$$\sigma_\alpha(\tau) = \Upsilon \left({}^{C}_{E_\alpha(+)}\mathrm{D}_0^{(\alpha)} \varepsilon_\alpha \right)(\tau). \tag{19}$$

(C2) The GFODE involving the Riemann-Liouville FD in the kernel of the positive-parametric MLF is given as:

$$\sigma_\alpha(\tau) = \Upsilon \left({}^{RL}_{E_\alpha(+)}\mathrm{D}_{+0}^{(\alpha)} \varepsilon_\alpha \right)(\tau). \tag{20}$$

(C3) The GFODE involving the Liouville-Caputo FD in the kernel of the negative-parametric MLF is suggested as:

$$\sigma_\alpha(\tau) = \Upsilon\left({}_{E_\alpha(-)}^{C}\mathrm{D}_0^{(\alpha)}\varepsilon_\alpha\right)(\tau). \tag{21}$$

(C4) The GFODE involving the Riemann-Liouville FD in the kernel of the negative-parametric MLF is

$$\sigma_\alpha(\tau) = \Upsilon\left({}_{E_\alpha(-)}^{RL}\mathrm{D}_0^{(\alpha)}\varepsilon_\alpha\right)(\tau). \tag{22}$$

We now call Eqs. (19), (20), (21), and (22) as the fractional-order spring-dashpot elements within the GFDs. They are used to structure the Voigt model (VM) and Maxwell model (MM) involving the Liouville-Caputo and Riemann-Liouville GFDs in the positive- and negative-parametric MLF kernels.

3.2. GENERAL CREEP AND RELAXATION REPRESENTATIONS WITHIN THE GFDS

Making use of the Boltzmann superposition principle and causal histories for $\tau \in \mathbb{R}_0^+$ (see, *e.g.*, [28]), we can present the creep and relaxation representations as follows.

The creep compliance and relaxation modulus functions are defined as [27]:

$$J_\alpha(\tau) = \frac{\varepsilon_\alpha(\tau)}{\sigma_\alpha(0)} \tag{23}$$

and

$$G_\alpha(\tau) = \frac{\sigma_\alpha(\tau)}{\varepsilon_\alpha(0)}, \tag{24}$$

respectively, where $\sigma_\alpha(0)$ is the initial stress condition, and $\varepsilon_\alpha(0)$ is the initial strain condition.

The creep and relaxation representations involving the Liouville-Caputo FD in the positive-parametric MLF kernel are given through the equations of the Volterra type:

$$\varepsilon_\alpha(\tau) = \sigma_\alpha(0)\, J_\alpha(\tau) + \int_0^\tau J_\alpha(\tau - t)\left({}_{E_\alpha(+)}^{C}\mathrm{D}_0^{(\alpha)}\sigma_\alpha\right)(t)\, dt \tag{25}$$

and

$$\sigma_\alpha(\tau) = \varepsilon_\alpha(0)\, G_\alpha(\tau) + \int_0^\tau G_\alpha(\tau - t)\left({}_{E_\alpha(+)}^{C}\mathrm{D}_0^{(\alpha)}\varepsilon_\alpha\right)(t)\, dt, \tag{26}$$

respectively.

The creep and relaxation representations involving the Riemann-Liouville FD in the positive-parametric MLF kernel are presented through the equations of the Volterra type:

$$\varepsilon_\alpha(\tau) = \sigma_\alpha(0)\, J_\alpha(\tau) + \int_0^\tau J_\alpha(\tau - t)\left({}_{E_\alpha(+)}^{RL}\mathrm{D}_0^{(\alpha)}\sigma_\alpha\right)(t)\, dt \tag{27}$$

and

$$\sigma_\alpha(\tau) = \varepsilon_\alpha(0)\, G_\alpha(\tau) + \int_0^\tau G_\alpha(\tau - t) \left({}^{RL}_{E_\alpha(+)}\mathrm{D}_0^{(\alpha)} \varepsilon_\alpha \right)(t)\, dt, \tag{28}$$

respectively.

The creep and relaxation representations involving the Riemann-Liouville FD in the negative-parametric MLF kernel are given through the equations of the Volterra type:

$$\varepsilon_\alpha(\tau) = \sigma_\alpha(0)\, J_\alpha(\tau) + \int_0^\tau J_\alpha(\tau - t) \left({}^{C}_{E_\alpha(-)}\mathrm{D}_0^{(\alpha)} \sigma_\alpha \right)(t)\, dt \tag{29}$$

and

$$\sigma_\alpha(\tau) = \varepsilon_\alpha(0)\, G_\alpha(\tau) + \int_0^\tau G_\alpha(\tau - t) \left({}^{RL}_{E_\alpha(-)}\mathrm{D}_0^{(\alpha)} \varepsilon_\alpha \right)(t)\, dt, \tag{30}$$

respectively.

The creep and relaxation representations involving the Riemann-Liouville FD in the negative-parametric MLF kernel are presented through the equations of the Volterra type:

$$\varepsilon_\alpha(\tau) = \sigma_\alpha(0)\, J_\alpha(\tau) + \int_0^\tau J_\alpha(\tau - t) \left({}^{RL}_{E_\alpha(-)}\mathrm{D}_0^{(\alpha)} \sigma_\alpha \right)(t)\, dt \tag{31}$$

and

$$\sigma_\alpha(\tau) = \varepsilon_\alpha(0)\, G_\alpha(\tau) + \int_0^\tau G_\alpha(\tau - t) \left({}^{RL}_{E_\alpha(-)}\mathrm{D}_0^{(\alpha)} \varepsilon_\alpha \right)(t)\, dt, \tag{32}$$

respectively.

3.3. THE VMS WITHIN THE GFDS

With the aid of the idea [21], the VM within the GFDs makes up of a HE and a GFODE in parallel.

The CEs of the general fractional-order VMs are as follows:

(D1) By virtue of Eqs. (25) and (26), the CE for the VM involving the Liouville-Caputo FD in the positive-parametric MLF kernel is written as:

$$\sigma_\alpha(\tau) = \Pi \varepsilon_\alpha(\tau) + \Upsilon \left({}^{C}_{E_\alpha(+)}\mathrm{D}_0^{(\alpha)} \varepsilon_\alpha \right)(\tau). \tag{33}$$

(D2) By using Eqs. (27) and (28), the CE for the VM involving the Riemann-Liouville FD in the positive-parametric MLF kernel is given as:

$$\sigma_\alpha(\tau) = \Pi \varepsilon_\alpha(\tau) + \Upsilon \left({}^{RL}_{E_\alpha(+)}\mathrm{D}_0^{(\alpha)} \varepsilon_\alpha \right)(\tau). \tag{34}$$

(D3) From Eqs. (29) and (30), the CE for the VM involving the Liouville-Caputo FD in the negative-parametric MLF kernel is given as:

$$\sigma_\alpha(\tau) = \Pi \varepsilon_\alpha(\tau) + \Upsilon \left({}^{C}_{E_\alpha(-)}\mathrm{D}_0^{(\alpha)} \varepsilon_\alpha \right)(\tau). \tag{35}$$

(D4) Making use of Eqs. (31) and (32), the CE for the VE involving the Riemann-Liouville FD in the negative-parametric MLF kernel is presented as:

$$\sigma_\alpha(\tau) = \Pi\varepsilon_\alpha(\tau) + \Upsilon\left({}_{E_\alpha(-)}^{RL}\mathrm{D}_0^{(\alpha)}\varepsilon_\alpha\right)(\tau). \tag{36}$$

3.4. CREEP BEHAVIORS OF THE GENERAL FRACTIONAL-ORDER VMS

The body is subjected to the one-step stress history

$$\sigma_\alpha(\tau) = \sigma_\alpha(0)\,\mathrm{H}(\tau), \tag{37}$$

where $\mathrm{H}(\tau)$ is the unit step function [27].

The corresponding creep differential equations (CDEs) for the general fractional-order VMs are given as follows.

From Eq. (33), the CDE for the general fractional-order VM can be given as:

$$\sigma_\alpha(0)\,\mathrm{H}(\tau) = \Pi\varepsilon_\alpha(\tau) + \Upsilon\left({}_{E_\alpha(+)}^{C}\mathrm{D}_{+0}^{(\alpha)}\varepsilon_\alpha\right)(\tau), \tag{38}$$

which yields the creep compliance function:

$$J_\alpha(\tau) = \frac{1}{\Pi}E_\alpha\left(\frac{\Pi}{\Pi+\Upsilon}\tau^\alpha\right) - \frac{1}{\Pi+\Upsilon}E_{\alpha,\alpha+1}\left(\frac{\Pi}{\Pi+\Upsilon}\tau^\alpha\right). \tag{39}$$

Making use of Eq. (34), the CDE for the general fractional-order VM is as follows:

$$\sigma_\alpha(0)\,\mathrm{H}(\tau) = \Pi\varepsilon_\alpha(\tau) + \Upsilon\left({}_{E_\alpha(+)}^{RL}\mathrm{D}_0^{(\alpha)}\varepsilon_\alpha\right)(\tau), \tag{40}$$

which leads to the creep compliance function:

$$J_\alpha(\tau) = \frac{1}{\Pi+\Upsilon}\left[E_\alpha\left(\frac{\Pi}{\Pi+\Upsilon}\tau^\alpha\right) - E_{\alpha,\alpha+1}\left(\frac{\Pi}{\Pi+\Upsilon}\tau^\alpha\right)\right]. \tag{41}$$

With the aid of Eq. (35), the CDE for the general fractional-order VM is presented as:

$$\sigma_\alpha(0)\,\mathrm{H}(\tau) = \Pi\varepsilon_\alpha(\tau) + \Upsilon\left({}_{E_\alpha(-)}^{C}\mathrm{D}_0^{(\alpha)}\varepsilon_\alpha\right)(\tau), \tag{42}$$

which reduces to the corresponding creep compliance function:

$$J_\alpha(\tau) = \frac{1}{\Pi}E_\alpha\left(-\frac{\Pi}{\Pi+\Upsilon}\tau^\alpha\right) + \frac{1}{\Pi+\Upsilon}E_{\alpha,\alpha+1}\left(-\frac{\Pi}{\Pi+\Upsilon}\tau^\alpha\right). \tag{43}$$

Making use of Eq. (36), the CDE for the general fractional-order VM is

$$\sigma_\alpha(0)\,\mathrm{H}(\tau) = \Pi\varepsilon_\alpha(\tau) + \Upsilon\left({}_{E_\alpha(-)}^{RL}\mathrm{D}_0^{(\alpha)}\varepsilon_\alpha\right)(\tau), \tag{44}$$

which deduces the creep compliance function:

$$J_\alpha(\tau) = \frac{1}{\Pi+\Upsilon}\left(E_\alpha\left(-\frac{\Pi}{\Pi+\Upsilon}\tau^\alpha\right) + E_{\alpha,\alpha+1}\left(-\frac{\Pi}{\Pi+\Upsilon}\tau^\alpha\right)\right) \tag{45}$$

The creep responses for the general fractional-order Voigt models are shown in Fig. 2.

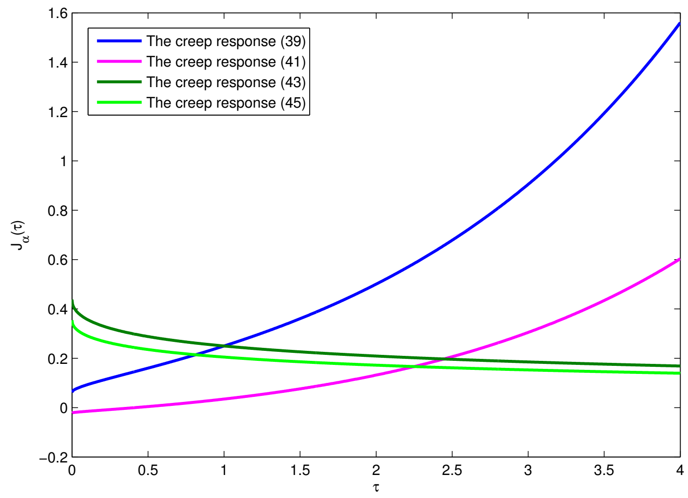

Fig. 2 – The plots of the creep responses for the Voigt models for the parameter $\alpha = 0.5$.

3.5. RELAXATION BEHAVIORS OF THE GENERAL FRACTIONAL-ORDER VMS

The body is subjected to the one-step strain history

$$\varepsilon_\alpha(\tau) = \varepsilon_\alpha(0)\,\mathrm{H}(\tau). \tag{46}$$

The relaxation differential equations (RDEs) for the general fractional-order VMs are presented as follows.

With the use of Eq. (46) and (33), the RDE for the general fractional-order VM can be rewritten as:

$$\sigma_\alpha(\tau) = \Pi\varepsilon_\alpha(0), \tag{47}$$

which leads to the relaxation modulus function:

$$G_\alpha(\tau) = \Pi. \tag{48}$$

Similarly, from Eqs. (46) and (34), the RDE for the general fractional-order VM can be presented as:

$$\sigma_\alpha(\tau) = \Pi\varepsilon_\alpha(0) + \Upsilon\varepsilon_\alpha(0)\,E_\alpha(\tau^\alpha), \tag{49}$$

which reduces to the relaxation modulus function:

$$G_\alpha(\tau) = \Pi + \Upsilon E_\alpha(\tau^\alpha). \tag{50}$$

From Eqs. (35) and (46), the RDE for the general fractional-order VM is rewritten as:

$$\sigma_\alpha(\tau) = \Pi\varepsilon_\alpha(0), \tag{51}$$

which implies the relaxation modulus function:

$$G_\alpha(\tau) = \Pi. \tag{52}$$

In virtue of Eqs. (46) and (36), the RDE for the general fractional-order VM is rewritten as:

$$\sigma_\alpha(\tau) = \Pi\varepsilon_\alpha(0) + \Upsilon\varepsilon_\alpha(0)E_\alpha(-\tau^\alpha), \tag{53}$$

which leads to the relaxation modulus function:

$$G_\alpha(\tau) = \Pi + \Upsilon E_\alpha(-\tau^\alpha). \tag{54}$$

The relaxation responses for the general fractional-order VMs are shown in Fig. 3.

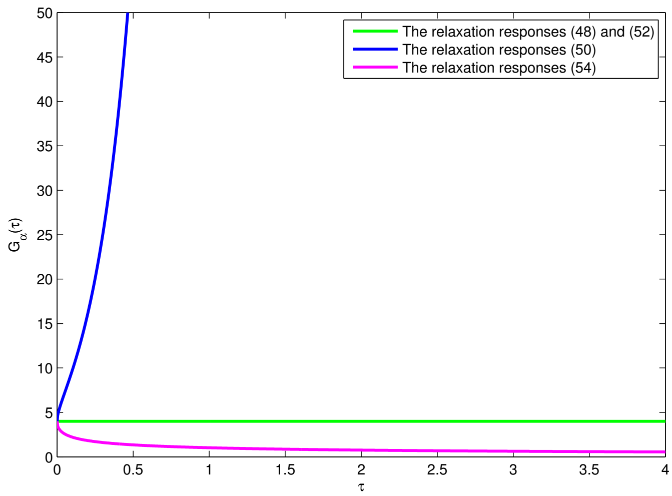

Fig. 3 – The plots of the relaxation responses for the VMs for the parameter $\alpha = 0.5$.

3.6. GENERAL FRACTIONAL-ORDER MMS WITHIN THE GFDS

In a similar way, the MM within the GFDs consists of a HE and a GFODE in series (see, for instance, [28]).

The CEs of the general fractional-order Maxwell models are as follows.

The MM involving the Liouville-Caputo FD in the positive-parametric MLF kernel, described by Eqs. (25) and (26), is written as:

$$\left({}^{C}_{E_\alpha(+)}\mathrm{D}_0^{(\alpha)}\varepsilon_\alpha\right)(\tau)=\frac{\sigma_\alpha(\tau)}{\Upsilon}+\frac{1}{\Pi}\left({}^{C}_{E_\alpha(+)}\mathrm{D}_0^{(\alpha)}\sigma_\alpha\right)(\tau). \tag{55}$$

Upon employing Eqs. (27) and (28), the MM involving the Riemann-Liouville FD in the positive-parametric MLF kernel is given as:

$$\left({}^{RL}_{E_\alpha(+)}\mathrm{D}_0^{(\alpha)}\varepsilon_\alpha\right)(\tau)=\frac{\sigma_\alpha(\tau)}{\Upsilon}+\frac{1}{\Pi}\left({}^{RL}_{E_\alpha(+)}\mathrm{D}_0^{(\alpha)}\sigma_\alpha\right)(\tau). \tag{56}$$

By using Eqs. (29) and (30), the MM involving the Liouville-Caputo FD in the negative-parametric MLF kernel is suggested as follows:

$$\left({}^{C}_{E_\alpha(-)}\mathrm{D}_0^{(\alpha)}\varepsilon_\alpha\right)(\tau)=\frac{\sigma_\alpha(\tau)}{\Upsilon}+\frac{1}{\Pi}\left({}^{C}_{E_\alpha(-)}\mathrm{D}_0^{(\alpha)}\sigma_\alpha\right)(\tau). \tag{57}$$

The MM involving the Riemann-Liouville FD in the negative-parametric MLF kernel, with the use of Eqs. (31) and (32), is presented as:

$$\left({}^{RL}_{E_\alpha(-)}\mathrm{D}_0^{(\alpha)}\varepsilon_\alpha\right)(\tau)=\frac{\sigma_\alpha(\tau)}{\Upsilon}+\frac{1}{\Pi}\left({}^{RL}_{E_\alpha(-)}\mathrm{D}_0^{(\alpha)}\sigma_\alpha\right)(\tau). \tag{58}$$

3.7. CREEP BEHAVIORS OF THE GENERAL FRACTIONAL-ORDER MMS

The body is subjected to the condition (37). In a similar procedure, the DEs for describing creep behaviors of the general fractional-order MMs are given as follows.

In view of Eq. (55), the CDE for the general fractional-order MM can be written as:

$$\left({}^{C}_{E_\alpha(+)}\mathrm{D}_0^{(\alpha)}\varepsilon_\alpha\right)(\tau)=\frac{\sigma_\alpha(0)}{\Upsilon}, \tag{59}$$

which implies the creep compliance function:

$$J_\alpha(\tau)=\frac{1}{\Upsilon}\left(1-\frac{\tau^\alpha}{\Gamma(1+\alpha)}\right)+\frac{1}{\Pi}. \tag{60}$$

In view of Eq. (56), the CDE for the general fractional-order MM can be given by:

$$\left({}^{RL}_{E_\alpha(+)}\mathrm{D}_0^{(\alpha)}\varepsilon_\alpha\right)(\tau)=\frac{\sigma_\alpha(0)}{\Upsilon}+\frac{\sigma_\alpha(0)}{\Pi}E_\alpha(\tau^\alpha), \tag{61}$$

which results in the creep compliance function:

$$J_\alpha(\tau)=\frac{1}{\Upsilon}\left(1-\frac{\tau^\alpha}{\Gamma(1+\alpha)}\right)+\frac{1}{\Pi}. \tag{62}$$

With the use of Eq. (57), the CDE for the general fractional-order MM can be written as:

$$\left({}^{C}_{E_\alpha(-)}\mathrm{D}_0^{(\alpha)}\varepsilon_\alpha\right)(\tau)=\frac{\sigma_\alpha(0)}{\Upsilon}, \tag{63}$$

which implies the creep compliance function:

$$J_\alpha(\tau) = \frac{1}{\Pi} + \frac{1}{\Upsilon}\left(1 + \frac{\tau^\alpha}{\Gamma(1+\alpha)}\right). \tag{64}$$

From Eq. (58), the CDE for the general fractional-order MM can be presented as:

$$\left({}^{RL}_{E_\alpha(-)}\mathrm{D}_0^{(\alpha)}\varepsilon_\alpha\right)(\tau) = \frac{\sigma_\alpha(0)}{\Upsilon} + \frac{\sigma_\alpha(0)}{\Pi}E_\alpha(-\tau^\alpha), \tag{65}$$

which yields the creep compliance function:

$$J_\alpha(\tau) = \frac{1}{\Pi} + \frac{1}{\Upsilon}\left(1 + \frac{\tau^\alpha}{\Gamma(1+\alpha)}\right). \tag{66}$$

The creep responses for the general fractional-order MMs are illustrated in Fig. 4.

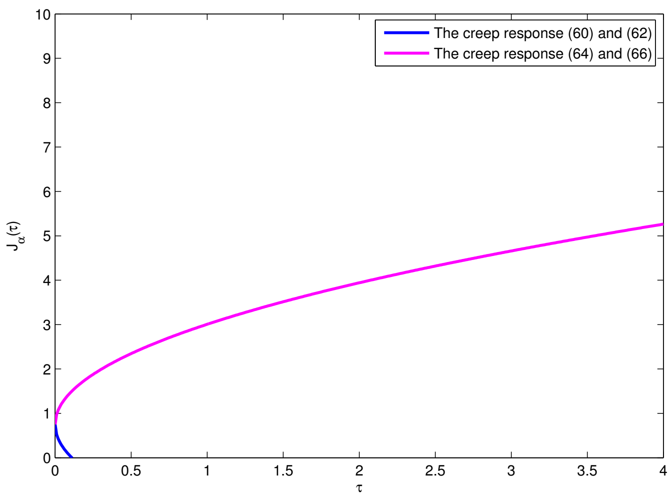

Fig. 4 – The plots of the creep responses for the MMs for the parameter $\alpha = 0.5$.

3.8. RELAXATION BEHAVIORS OF THE GENERAL FRACTIONAL-ORDER MMS

The body is subjected to the condition (46). In a similar way, the DEs for describing relaxation behaviors of the general fractional-order MMs are presented as follows.

Making use of Eq. (46), the RDE for the general fractional-order MM takes the form:

$$\frac{\sigma_\alpha(\tau)}{\Upsilon} + \frac{1}{\Pi}\left({}^{C}_{E_\alpha(+)}\mathrm{D}_0^{(\alpha)}\sigma_\alpha\right)(\tau) = 0, \tag{67}$$

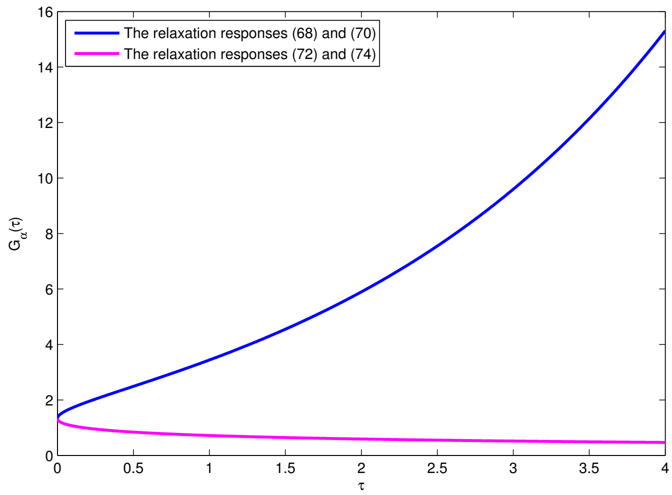

Fig. 5 – The plots of the relaxation responses for the MMs for the parameter $\alpha = 0.5$.

which leads to the relaxation modulus function:

$$G_\alpha(\tau) = \frac{\Upsilon\Pi}{\Upsilon+\Pi} E_\alpha\left(\frac{\Pi}{\Upsilon+\Pi}\tau^\alpha\right). \tag{68}$$

With the use of Eq. (56), the RDE for the general fractional-order MM is expressed by:

$$\varepsilon_\alpha(0) E_\alpha(\tau^\alpha) = \frac{\sigma_\alpha(\tau)}{\Upsilon} + \frac{1}{\Pi}\left({}^{RL}_{E_\alpha(+)}\mathrm{D}_0^{(\alpha)}\sigma_\alpha\right)(\tau), \tag{69}$$

which yields the relaxation modulus function:

$$G_\alpha(\tau) = \frac{\Upsilon\Pi}{\Upsilon+\Pi} E_\alpha\left(\frac{\Pi}{\Upsilon+\Pi}\tau^\alpha\right). \tag{70}$$

From Eq. (57), the RDE for the general fractional-order MM is represented as:

$$\frac{\sigma_\alpha(\tau)}{\Upsilon} + \frac{1}{\Pi}\left({}^{C}_{E_\alpha(-)}\mathrm{D}_0^{(\alpha)}\sigma_\alpha\right)(\tau) = 0, \tag{71}$$

which implies the relaxation modulus function:

$$G_\alpha(\tau) = \frac{\Upsilon\Pi}{\Upsilon+\Pi} E_\alpha\left(-\frac{\Pi}{\Upsilon+\Pi}\tau^\alpha\right). \tag{72}$$

Owing to Eq. (58), the RDE for the general fractional-order MM is given by:

$$\varepsilon_\alpha(0) E_\alpha(-\tau^\alpha) = \frac{\sigma_\alpha(\tau)}{\Upsilon} + \frac{1}{\Pi}\left({}^{RL}_{E_\alpha(+)}\mathrm{D}_0^{(\alpha)}\sigma_\alpha\right)(\tau), \tag{73}$$

which leads to the relaxation modulus function:

$$G_\alpha(\tau) = \frac{\Upsilon\Pi}{\Upsilon+\Pi} E_\alpha\left(-\frac{\Pi}{\Upsilon+\Pi}\tau^\alpha\right). \tag{74}$$

The relaxation responses for the general fractional-order MMs are displayed in Fig. 5.

4. CONCLUSION

In the present work, we proposed the general fractional-order Maxwell and Voigt models within the kernels of the MLFs *via* the general laws of deformation, for the first time to our knowledge. The creep and relaxation characteristics for the general fractional-order rheological models were presented in detail. This may provide a new perspective for describing the complex behaviors of the general fractional-order linear viscoelasticity with memory effect.

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