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COMPOSITE BERNOULLI-LAGUERRE COLLOCATION METHOD FOR A CLASS OF HYPERBOLIC TELEGRAPH-TYPE EQUATIONS

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Abstract. In this work, we introduce an efficient Bernoulli-Laguerre collocation method for solving a class of hyperbolic telegraph-type equations in one dimension. Bernoulli and Laguerre polynomials and their properties are utilized to reduce the aforementioned problems to systems of algebraic equations. The proposed collocation method, both in spatial and temporal discretizations, is successfully developed to handle the two-dimensional case. In order to highlight the effectiveness of our approachs, several numerical examples are given. The approximation techniques and results developed in this paper are appropriate for many other problems on multiple-dimensional domains, which are not of standard types.

Key words: Hyperbolic telegraph-type equations, Bernoulli-Laguerre collocation method, Gauss-Bernoulli nodes, Gauss-Laguerre nodes.

1. INTRODUCTION

The hyperbolic-type partial differential equations (PDEs) is a significant class of the PDEs because of their appearance in many fields of science and engineering such as atomic physics, aerospace, industry, biology, and engineering problems such as vibrations of structures, beams and buildings [1, 2]. Telegraph equations are hyperbolic PDEs that are applicable in several fields such as wave propagation [3], random walk theory [4], and signal analysis [5].

We recall that the numerical methods for solving PDEs play a crucial role in science and engineering because they are encountered in mathematical modeling [6–11]. In recent years, several numerical approaches have been introduced to numerically approximate the solution of hyperbolic telegraph equations such as collocation approach based on radial basis functions [12], Chebyshev Tau method [13, 14], Jacobi Tau method [15], Legendre multi-wavelet Galerkin method [16], homotopy perturbation method [17], variational iteration method [18], spectral collocation method [19, 20], differential quadrature method [21], Haar wavelet method [22], dual reciprocity boundary integral equation method [23], and variational iteration method [24]; see also other relevant recent papers in the broad area of differential equations and PDEs and their applications to the description of a series of physical phenomena [25–31].

Numerical solutions of both linear and nonlinear one-dimensional hyperbolic telegraph equation have been reported, *e.g.*, in [32, 33]. Also, the unconditionally stable finite difference schemes was reported in [34, 35]. Dehghan and Lakestani [36] utilized the Chebyshev cardinal functions and Saadatmandi and Dehghan [14] utilized the Chebyshev Tau method for expanding the approximate solution of one-dimensional telegraph equation. Mohebbi and Dehghan [37] introduced a higher order compact finite difference approximation of fourth order in space and made use of the collocation method in time direction. We recall that other techniques utilized for numerical solutions of one-dimensional (1D) hyperbolic telegraph equation generalized Laguerre-Gauss-Radau scheme [38] and collocation scheme technique [39].

The article is structured as follows. Section 2 introduces some characteristics of Bernoulli and Laguerre polynomials. Section 3 is dedicated to the hypothetical determination of the Bernoulli-Laguerre collocation method (B-LCM) for solving 1D hyperbolic telegraph type equations with variable coefficients. Section 4 is dedicated to applying the B-LCM for two-dimensional (2D) hyperbolic telegraph type equations with variable coefficients. In addition, in Section 5 the proposed technique is connected to various test issues. Finally, some concluding remarks are given in Section 6.

2. PRELIMINARIES

In this Section, we collect some important properties of Bernoulli and Laguerre polynomials.

2.1. BERNOULLI POLYNOMIALS

We recall that the generating function of the Bernoulli polynomials is expressed by

$$\frac{te^{xt}}{e^t-1} = e^{t\mathcal{B}(x)} = \sum_{k=0}^{\infty} \mathcal{B}_k(x)\frac{t^k}{k!}, \tag{1}$$

when $x = 0$, $\mathcal{B}_k(0) = \mathcal{B}_k$ are called the k-th Bernoulli numbers. Taking into account (1), we conclude that (see [40])

$$\mathcal{B}_0 = 1,\ (\mathcal{B}+1)^k - \mathcal{B}^k = \mathcal{B}_k(1) - \mathcal{B}_k = \delta_{1,k}, \tag{2}$$

where $\delta_{m,n}$ is the Kronecker symbol.

The Bernoulli polynomials $\mathcal{B}_k(x)$ of degree k satisfy the following relations

$$\mathcal{B}_k(x) = \sum_{l=0}^{k} \begin{pmatrix} k \\ l \end{pmatrix} \mathcal{B}_k x^{k-l}. \tag{3}$$

$$\sum_{l=0}^{k} \begin{pmatrix} k+1 \\ l \end{pmatrix} \mathcal{B}_l(x) = (1-k)x^k,\ k = 0,1,\ldots. \tag{4}$$

$$\int_0^1 B_i(x)B_j(x)dx = (-1)^{i-1}\frac{j!i!}{(j+i)!}B_{j+i},\ j,i \geq 1. \tag{5}$$

2.2. LAGUERRE POLYNOMIALS

Now, let $\Lambda = (0,\infty)$ and $w^{(\alpha)}(x) = e^{-x}$ be a weight function on Λ in the usual sense [41, 42]. Define

$$L^2_w(\Lambda) = \{v \mid v \text{ is measurable on } \Lambda \text{ and } \ ||v||_w < \infty\},$$

equipped with the following inner product and norm

$$(u,v)_w = \int_\Lambda u(x)\ v(x)\ w(x)\ dx, \qquad ||v||_w = (u,v)_w^{\frac{1}{2}}.$$

Next, let $\Lambda = (0,\infty),\ w(x) = e^{-x},$ and $\mathcal{L}_\ell(x)$ be the Laguerre polynomial of degree ℓ, defined by

$$\mathcal{L}_\ell(x) = \frac{1}{\ell!}e^x\partial_x^\ell(x^\ell e^{-x}), \quad \ell = 0,1,\ldots. \tag{6}$$

They satisfy the equations

$$\partial_x(x\,e^{-x}\partial_x\mathcal{L}_\ell(x))+\ell e^{-x}\mathcal{L}_\ell(x)=0 \quad x\in\Lambda,$$

and

$$\mathcal{L}_\ell(x)=\partial_x\mathcal{L}_\ell(x)-\partial_x\mathcal{L}_{\ell+1}(x), \quad \ell\geq 0.$$

The set of Laguerre polynomials is the $L^2_w(\Lambda)$-orthogonal system, namely,

$$\int_\Lambda \mathcal{L}_j(x)\mathcal{L}_k(x)w(x)dx=\delta_{jk}, \quad \forall i,j\geq 0, \tag{7}$$

where δ_{jk} is the Kronecher function.

3. DISCRETIZATION IN 1D

In this Section, we introduce the B-LCM for 1D linear and nonlinear hyperbolic telegraph-type equations (HTTEs) with the homogeneous and nonhomogeneous conditions.

3.1. LINEAR HTTES

Here, we consider the B-LCM for solving linear 1D HTTEs with variable coefficients:

$$\begin{gathered}\frac{\partial^2 v(x,t)}{\partial t^2}+2\rho(x,t)\frac{\partial v(x,t)}{\partial t}+\varrho^2(x,t)v(x,t)=\sigma(x,t)\frac{\partial^2 v(x,t)}{\partial x^2}+\chi(x,t),\\ 0\leq x\leq 1,\, 0\leq t<\infty,\end{gathered} \tag{8}$$

with the homogeneous conditions

$$v(0,t)=v(1,t)=v(x,0)=\frac{\partial v(x,t)}{\partial t}|_{t=0}=0, \tag{9}$$

For $\rho>0, \varrho=0$, Eq. (8) represents a damped wave equation, and if $\rho>\varrho>0$, it is known as telegraph equation.

The point of our method to get solution can be extended, using combination of basis functions of Bernoulli and Laguerre polynomials, in the form

$$v(x,t)\backsimeq\sum_{i=0}^{N-2}\sum_{j=0}^{M-2}c_{i,j}\phi_i(x)\varphi_j(t)=\psi(x,t)C, \tag{10}$$

where, $c_{i,j},\ i=0,1,\ldots,N-2,\ j=0,1,\ldots,M-2$ are the unknown coefficients,

$$C=[c_{0,0},c_{1,0},\ldots,c_{N-2,0};c_{0,1},c_{1,1},\ldots,c_{N-2,1};c_{0,M-2},c_{1,M-2},\ldots,c_{N-2,M-2}]^T,$$

N and M are any arbitrary positive integers, and

$$\phi_i(x)=x(1-x)\mathcal{B}_i(x), \tag{11}$$

$$\varphi_j(t) = t^2\mathcal{L}_j(t). \tag{12}$$

Also $\psi(x,t)$ is a $1\times(N-1)(M-1)$ matrix introduced as follows

$$\begin{aligned}\psi(x,t) =&[r_{0,0}(x,t), r_{1,0}(x,t), \dots, r_{N-2,0}(x,t); r_{0,1}(x,t), r_{1,1}(x,t), \dots, r_{N-2,1}(x,t)\\ &; r_{0,M-2}(x,t), r_{1,M-2}(x,t), \dots, r_{N-2,M-2}(x,t)],\end{aligned}$$

where

$$r_{i,j}(x,t) = \phi_i(x)\varphi_j(t),\ i = 0,1,\dots,N-2,\ j = 0,1,\dots,M-2.$$

Substituting Eq. (10) into Eq. (8) yields:

$$\begin{aligned}&\frac{\partial^2}{\partial t^2}(\sum_{i=0}^{N-2}\sum_{j=0}^{M-2} c_{i,j}\phi_i(x)\varphi_j(t)) + 2\rho(x,t)\frac{\partial}{\partial t}(\sum_{i=0}^{N-2}\sum_{j=0}^{M-2} c_{i,j}\phi_i(x)\varphi_j(t))\\ &+\varrho^2(x,t)\sum_{i=0}^{N-2}\sum_{j=0}^{M-2} c_{i,j}\phi_i(x)\varphi_j(t) = \sigma(x,t)\frac{\partial^2}{\partial x^2}(\sum_{i=0}^{N-2}\sum_{j=0}^{M-2} c_{i,j}\phi_i(x)\varphi_j(t)) + \chi(x,t).\end{aligned} \tag{13}$$

Assume that:

$$\begin{aligned}f_{i,j}(x,t) =&\frac{\partial^2}{\partial t^2}(\phi_i(x)\varphi_j(t)) + 2\rho(x,t)\frac{\partial}{\partial t}(\phi_i(x)\varphi_j(t)) + \varrho^2(x,t)\phi_i(x)\varphi_j(t)\\ &-\sigma(x,t)\frac{\partial^2}{\partial x^2}(\phi_i(x)\varphi_j(t)),\end{aligned}$$

at that point, Eq. (13) can be modified as:

$$\sum_{i=0}^{N-2}\sum_{j=0}^{M-2} c_{i,j} f_{i,j}(x,t) = \chi(x,t). \tag{14}$$

Collocating Eq. (14) in $N-1$ and $M-1$ roots of the Bernoulli polynomial $\mathcal{B}_{N-1}(x)$ and Laguerre polynomial $\mathcal{L}_{M-1}(t)$, respectively, the Gauss-Bernoulli in combination with Gauss-Laguerre nodes, we obtain:

$$\begin{aligned}\sum_{i=0}^{N-2}\sum_{j=0}^{M-2} c_{i,j} f_{i,j}(x_{n,i}, t_{m,j}) = \chi(x_{n,i}, t_{m,j}),\ &for\ n = 0,1,\dots,N-2,\\ &m = 0,1,\dots,M-2,\end{aligned} \tag{15}$$

which can be written in the following matrix form:

$$\mathbf{F}^T C = \mathbf{R},$$

where

$$\mathbf{R} = [\chi(x_{n,0},t_{m,0}),\chi(x_{n,1},t_{m,0}),\ldots,\chi(x_{n,N-2},t_{m,0});\chi(x_{n,0},t_{m,1}),\chi(x_{n,1},t_{m,1}),\ldots,$$
$$\chi(x_{n,N-2},t_{m,1});\chi(x_{n,0},t_{m,M-2}),\chi(x_{n,1},t_{m,M-2}),\ldots,\chi(x_{n,N-2},t_{m,M-2})]^T,$$

and

$$\mathbf{F} = (f_{ijnm}),\ i,n = 0,1,\ldots,N-2,\ j,m = 0,1,\ldots,M-2,$$

in which the elements of the matrix **F** are determined as follows:

$$f_{ijnm} = f_{i,j}(x_{n,i},t_{m,j}),\ i,n = 0,1,\ldots,N-2,\ j,m = 0,1,\ldots,M-2.$$

Finally, the unknown vector C can be computed by:

$$C = (\mathbf{F}^T)^{-1}\mathbf{R}.$$

In this manner, the approximate solution of Eq. (8) is given by $v(x,t) = \psi(x,t)C$.

3.2. NONLINEAR HTTES

The 1D nonlinear hyperbolic telegraph-type equations with variable coefficients is given by:

$$\frac{\partial^2 v(x,t)}{\partial t^2} + 2\rho(x,t)\frac{\partial v(x,t)}{\partial t} - \sigma(x,t)\frac{\partial^2 v(x,t)}{\partial x^2} = \chi(x,t,v),\ 0 \le x \le 1,\ 0 \le t < \infty, \tag{16}$$

with the homogeneous conditions

$$v(0,t) = v(1,t) = v(x,0) = \frac{\partial v(x,t)}{\partial t}|_{t=0} = 0, \tag{17}$$

We first approximate the solution $v(x,t)$ using Eq. (10); then we get

$$\frac{\partial^2 \psi(x,t)C}{\partial t^2} + 2\rho(x,t)\frac{\partial \psi(x,t)C}{\partial t} - \sigma(x,t)\frac{\partial^2 \psi(x,t)C}{\partial x^2} = \chi(x,t,\psi(x,t)C), \tag{18}$$

Now, we collocate Eq. (18) at points $(x_{n,i},t_{m,j})$ for $n = 0,1,\ldots,N-2,\ m = 0,1,\ldots,M-2$. Hence, we have

$$\begin{aligned}\frac{\partial^2 \psi(x_{n,i},t_{m,j})C}{\partial t^2} &+ 2\rho(x_{n,i},t_{m,j})\frac{\partial \psi(x_{n,i},t_{m,j})C}{\partial t} - \sigma(x_{n,i},t_{m,j})\frac{\partial^2 \psi(x_{n,i},t_{m,j})C}{\partial x^2} \\ &= \chi(x_{n,i},t_{m,j},\psi(x_{n,i},t_{m,j})C),\end{aligned} \tag{19}$$

for $n = 0,1,\ldots,N-2,\ m = 0,1,\ldots,M-2$. Equation (19) generates a nonlinear system of equations that can be solved to obtain the unknown vector C.

3.3. TREATMENT OF THE NONHOMOGENEOUS CONDITIONS

Next we can simply alter the right-hand side to deal with the nonhomogeneous initial-boundary conditions. We consider, for example, the 1D linear and nonlinear hyperbolic telegraph-type equations (8) and (16) subject to the nonhomogeneous initial-boundary conditions:

$$\begin{aligned} v(0,t) =& q_0(t), v(1,t) = q_1(t), \quad 0 \le t < \infty \\ v(x,0) =& p_0(x), \frac{\partial v(x,t)}{\partial t}|_{t=0} = p_1(x), \quad 0 \le x \le 1, \end{aligned} \tag{20}$$

where χ, q_0, q_1, p_0, p_1 are known functions and the function v is unknown.

Presently, we assume the accompanying transformation

$$V(x,t) = v(x,t) + a_0(t) + a_1(t)x + x(x-1)(b_0(x) + b_1(x)t) \tag{21}$$

where

$$a_0(t) = -q_0(t), \quad a_1(t) = q_0(t) - q_1(t), \quad b_0(x) = \frac{(1-x)q_0(0) + q_1(0)x - p_0(x)}{x(x-1)},$$

$$b_1(x) = \frac{\frac{\partial q_0(t)}{\partial t}|_{t=0} - p_1(x) - x(\frac{\partial q_0(t)}{\partial t}|_{t=0} - \frac{\partial q_1(t)}{\partial t}|_{t=0})}{x(x-1)}.$$

The transformation (21) turns the nonhomogeneous initial-boundary conditions (20) into the homogeneous initial-boundary conditions

$$V(0,t) = V(1,t) = V(x,0) = \frac{\partial V(x,t)}{\partial t}|_{t=0} = 0, \quad 0 \le t < \infty, \quad 0 \le x \le 1. \tag{22}$$

Subsequently it suffices to solve the following linear hyperbolic telegraph-type equations:

$$\begin{aligned} \frac{\partial^2 V(x,t)}{\partial t^2} &+ 2\rho(x,t)\frac{\partial V(x,t)}{\partial t} + \varrho^2(x,t)V(x,t) = \sigma(x,t)\frac{\partial^2 V(x,t)}{\partial x^2} + \mathcal{G}(x,t), \\ &, 0 \le x \le 1, t > 0, \end{aligned} \tag{23}$$

and nonlinear hyperbolic telegraph-type equations:

$$\frac{\partial^2 V(x,t)}{\partial t^2} + 2\rho(x,t)\frac{\partial V(x,t)}{\partial t} - \sigma(x,t)\frac{\partial^2 V(x,t)}{\partial x^2} = \mathcal{G}(x,t,V), 0 \le x \le 1, t > 0. \tag{24}$$

4. DISCRETIZATION IN 2D

In this Section, we use the B-LCM for linear and nonlinear 2D hyperbolic telegraph-type equations with variable coefficients with the homogeneous and nonhomogeneous conditions.

4.1. LINEAR HTTES

We consider the following 2D linear HTTE

$$\frac{\partial^2 v(x,y,t)}{\partial t^2}+2\rho(x,y,t)\frac{\partial v(x,y,t)}{\partial t}+\varrho^2(x,y,t)v(x,y,t)=\sigma(x,y,t)(\frac{\partial^2 v(x,y,t)}{\partial x^2}+\frac{\partial^2 v(x,y,t)}{\partial y^2})+\chi(x,y,t),\, 0\le x\le 1,\, 0\le y\le 1,\, 0\le t<\infty, \tag{25}$$

with the homogeneous conditions

$$v(0,y,t)=v(1,y,t)=v(x,0,t)=v(x,1,t)=v(x,y,0)=\frac{\partial v(x,y,t)}{\partial t}|_{t=0}=0. \tag{26}$$

The point of our method to get solution can be extended, using combination of basis functions of Bernoulli and Laguerre polynomials, in the form

$$u(x,y,t)\backsimeq\sum_{i=0}^{N-2}\sum_{j=0}^{M-2}\sum_{k=0}^{K-2}c_{i,j,k}\phi_i(x)\varphi_j(y)\psi_k(t)=F(x,y,t)C, \tag{27}$$

where $c_{i,j,k},\ i=0,1,\ldots,N-2,\ j=0,1,\ldots,M-2,\ k=0,1,\ldots,K-2$ are the unknown coefficients,

$$C=[c_{0,0,0},c_{0,0,1},\ldots,c_{0,0,K-2},c_{0,1,0},c_{0,1,1},\ldots,c_{0,1,K-2},\ldots,c_{0,M-2,K-2};c_{1,0,0},\\ c_{1,0,1},\ldots,c_{1,0,K-2},c_{1,1,0},c_{1,1,1},\ldots,c_{1,1,K-2},\ldots,c_{1,M-2,K-2};c_{N-2,0,0},\\ c_{N-2,0,1},\ldots,c_{N-2,0,K-2},c_{N-2,1,0},c_{N-2,1,1},\ldots,c_{N-2,1,K-2},\ldots,\\ c_{N-2,M-2,K-2}]^T,$$

$N,\ M$, and K are any arbitrary positive integers, and

$$\phi_i(x)=x(1-x)\mathcal{B}_i(x), \tag{28}$$

$$\varphi_j(y)=y(1-y)\mathcal{B}_j(y), \tag{29}$$

$$\psi_k(t)=t^2\mathcal{L}_k(t). \tag{30}$$

Also $\digamma(x,y,t)$ is $1\times(N-1)(M-1)(K-1)$ matrix introduced as follows

$$\begin{aligned}\digamma(x,y,t) =&[r_{0,0,0}(x,y,t), r_{0,0,1}(x,y,t),\ldots, r_{0,0,K-2}(x,y,t), r_{0,1,0}(x,y,t),\\ & r_{0,1,1}(x,y,t),\ldots, r_{0,M-2,K-2}(x,y,t); r_{1,0,0}(x,y,t), r_{1,0,1}(x,y,t),\ldots,\\ & r_{1,0,K-2}(x,y,t), r_{1,1,0}(x,y,t), r_{1,1,1}(x,y,t),\ldots, r_{1,1,K-2}(x,y,t),\ldots,\\ & r_{1,M-2,K-2}(x,y,t); r_{N-2,0,0}(x,y,t), r_{N-2,0,1}(x,y,t),\ldots,\\ & r_{N-2,0,K-2}(x,y,t), r_{N-2,1,0}(x,y,t), r_{N-2,1,1}(x,y,t),\\ & \ldots, r_{N-2,1,K-2}(x,y,t),\ldots, r_{N-2,M-2,K-2}(x,y,t)],\end{aligned}$$

where

$$r_{i,j,k}(x,y,t) = \phi_i(x)\varphi_j(y)\,\psi_k(t),\ i=0,1,\ldots,N-2,\ j=0,1,\ldots,M-2,\\ k=0,1,\ldots,K-2.$$

Substituting Eq. (27) into Eq. (25) yields:

$$\begin{aligned}&\frac{\partial^2}{\partial t^2}\Big(\sum_{i=0}^{N-2}\sum_{j=0}^{M-2}\sum_{k=0}^{K-2} c_{i,j,k}\phi_i(x)\varphi_j(y)\psi_k(t)\Big)+2\rho(x,y,t)\frac{\partial}{\partial t}\\ &\times\Big(\sum_{i=0}^{N-2}\sum_{j=0}^{M-2}\sum_{k=0}^{K-2} c_{i,j,k}\phi_i(x)\varphi_j(y)\psi_k(t)\Big)+\varrho^2(x,y,t)\\ &\times\sum_{i=0}^{N-2}\sum_{j=0}^{M-2}\sum_{k=0}^{K-2} c_{i,j,k}\phi_i(x)\varphi_j(y)\psi_k(t)=\sigma(x,y,t)\\ &\times\frac{\partial^2}{\partial x^2}\Big(\sum_{i=0}^{N-2}\sum_{j=0}^{M-2}\sum_{k=0}^{K-2} c_{i,j,k}\phi_i(x)\varphi_j(y)\psi_k(t)\Big)+\sigma(x,y,t)\\ &\times\frac{\partial^2}{\partial y^2}\Big(\sum_{i=0}^{N-2}\sum_{j=0}^{M-2}\sum_{k=0}^{K-2} c_{i,j,k}\phi_i(x)\varphi_j(y)\psi_k(t)\Big)+\chi(x,y,t).\end{aligned}\tag{31}$$

Assume that:

$$\begin{aligned}f_{i,j,k}(x,y,t)=&\frac{\partial^2}{\partial t^2}(\phi_i(x)\varphi_j(y)\psi_k(t))+2\rho(x,y,t)\frac{\partial}{\partial t}(\phi_i(x)\varphi_j(y)\psi_k(t))\\ &+\varrho^2(x,y,t)\phi_i(x)\varphi_j(y)\psi_k(t)-\sigma(x,y,t)\frac{\partial^2}{\partial x^2}(\phi_i(x)\varphi_j(y)\psi_k(t))\\ &-\sigma(x,y,t)\frac{\partial^2}{\partial y^2}(\phi_i(x)\varphi_j(y)\psi_k(t)),\end{aligned}$$

at that point, Eq. (31) can be modified as:

$$\sum_{i=0}^{N-2}\sum_{j=0}^{M-2}\sum_{k=0}^{K-2} c_{i,j,k}f_{i,j,k}(x,y,t)=\chi(x,y,t).\tag{32}$$

Collocating Eq. (32) in $N-1,\ M-1$, and $K-1$ roots of the Bernoulli polynomials $\mathcal{B}_{N-1}(x)$, $\mathcal{B}_{M-1}(y)$ and Laguerre polynomial $\mathcal{L}_{K-1}(t)$, respectively, the Gauss-Bernoulli in combination with Gauss-Laguerre nodes, we obtain

$$\sum_{i=0}^{N-2}\sum_{j=0}^{M-2}\sum_{k=0}^{K-2} c_{i,j,k} f_{i,j,k}(x_{n,i}, y_{m,j}, t_{l,k}) = \chi(x_{n,i}, y_{m,j}, t_{l,k}),\ for\ n = 0,1,\ldots,N-2,$$
$$m = 0,1,\ldots,M-2,\ l = 0,1,\ldots,K-2. \tag{33}$$

which can be written in the following matrix form:

$$\mathbf{F}^T C = \mathbf{R},$$

where

$$\begin{aligned}\mathbf{R} =&[\chi(x_{n,0}, y_{m,0}, t_{l,0}), \chi(x_{n,0}, y_{m,0}, t_{l,1}), \ldots, \chi(x_{n,0}, y_{m,0}, t_{l,K-2}), \chi(x_{n,0}, y_{m,1}, t_{l,0}),\\ &\chi(x_{n,0}, y_{m,1}, t_{l,1}), \ldots, \chi(x_{n,0}, y_{m,1}, t_{l,K-2}), \ldots, \chi(x_{n,0}, y_{m,M-2}, t_{l,K-2});\\ &\chi(x_{n,1}, y_{m,0}, t_{l,0}), \chi(x_{n,1}, y_{m,0}, t_{l,1}), \ldots, \chi(x_{n,1}, y_{m,0}, t_{l,K-2}), \chi(x_{n,1}, y_{m,1} t_{l,0}),\\ &\chi(x_{n,1}, y_{m,1}, t_{l,1}), \ldots, \chi(x_{n,1}, y_{m,1}, t_{l,K-2}), \ldots, \chi(x_{n,1}, y_{m,M-2}, t_{l,K-2});\\ &\chi(x_{n,N-2}, y_{m,0}, t_{l,0}), \chi(x_{n,N-2}, y_{m,0}, t_{l,1}), \ldots, \chi(x_{n,N-2}, y_{m,0}, t_{,l,K-2}),\\ &\chi(x_{n,N-2}, y_{m,1}, t_{l,0}), \chi(x_{n,N-2}, y_{m,1}, t_{l,1}), \ldots, \chi(x_{n,N-2}, y_{m,1}, t_{l,K-2}),\\ &\ldots, \chi(x_{n,N-2}, y_{m,M-2}, t_{l,K-2})]^T,\end{aligned}$$

and

$$\mathbf{F} = (f_{ijknml}),\ i,n = 0,1,\ldots,N-2,\ j,m = 0,1,\ldots,M-2,\ k,l = 0,1,\ldots,K-2,$$

in which the elements of the matrix **F** are determined as follows:

$$f_{ijknml} = f_{i,j,k}(x_{n,i}, y_{m,j}, t_{l,k}),\ i,n = 0,1,\ldots,N-2,\ j,m = 0,1,\ldots,M-2,$$
$$k,l = 0,1,\ldots,K-2.$$

Finally, the unknown vector C can be computed by:

$$C = (\mathbf{F}^T)^{-1}\mathbf{R}.$$

In this manner, the approximate solution of Eq. (25) is given by $v(x,y,t) = F(x,y,t)C$.

4.2. NONLINEAR HTTES

The 2D nonlinear hyperbolic telegraph-type equation with variable coefficients is given by:

$$\begin{aligned}\frac{\partial^2 v(x,y,t)}{\partial t^2}+2\rho(x,y,t)\frac{\partial v(x,y,t)}{\partial t}-\sigma(x,y,t)\left(\frac{\partial^2 v(x,y,t)}{\partial x^2}+\frac{\partial^2 v(x,y,t)}{\partial y^2}\right)\\ =\chi(x,y,t,v),\, 0\le x\le 1,\, 0\le y\le 1,\, 0\le t<\infty,\end{aligned} \tag{34}$$

with the homogeneous conditions

$$v(0,y,t)=v(1,y,t)=v(x,0,t)=v(x,1,t)=v(x,y,0)=\frac{\partial v(x,y,t)}{\partial t}|_{t=0}=0, \tag{35}$$

We first approximate the solution $v(x,y,t)$ using Eq. (27); then we get

$$\begin{aligned}\frac{\partial^2 F(x,y,t)C}{\partial t^2}+2\rho(x,y,t)\frac{\partial F(x,y,t)C}{\partial t}-\sigma(x,y,t)\left(\frac{\partial^2 F(x,y,t)C}{\partial x^2}+\frac{\partial^2 F(x,y,t)C}{\partial y^2}\right)\\ =\chi(x,y,t,F(x,y,t)C),\, 0\le x\le 1,\, 0\le y\le 1,\, 0\le t<\infty,\end{aligned} \tag{36}$$

Now, we collocate Eq. (36) at points $(x_{n,i},y_{m,j},t_{l,k})$ for $n=0,1,\ldots,N-2$, $m=0,1,\ldots,M-2$, $l=0,1,\ldots,K-2$. Hence, we have

$$\begin{aligned}&\frac{\partial^2 F(x_{n,i},y_{m,j},t_{l,k})C}{\partial t^2}+2\rho(x_{n,i},y_{m,j},t_{l,k})\frac{\partial F(x_{n,i},y_{m,j},t_{l,k})C}{\partial t}\\ &-\sigma(x_{n,i},y_{m,j},t_{l,k})\left(\frac{\partial^2 F(x_{n,i},y_{m,j},t_{l,k})C}{\partial x^2}+\frac{\partial^2 F(x_{n,i},y_{m,j},t_{l,k})C}{\partial y^2}\right)\\ &=\chi(x_{n,i},y_{m,j},t_{l,k},F(x_{n,i},y_{m,j},t_{l,k})C).\end{aligned} \tag{37}$$

Equation (37) generates a nonlinear system of equations that can be solved to obtain the unknown vector C.

4.3. TREATMENT OF THE NONHOMOGENEOUS CONDITIONS

We can simply alter the right-hand side to deal with the nonhomogeneous initial-boundary conditions. We consider, for example, the 2D linear and nonlinear hyperbolic telegraph-type equations with variable coefficients (25) and (34) subject to the nonhomogeneous initial-boundary conditions:

$$\begin{aligned}u(0,y,t)=&q_0(y,t),\, u(1,y,t)=q_1(y,t), \quad 0\le y\le 1, 0\le t<\infty,\\ u(x,0,t)=&q_2(x,t),\, u(x,1,t)=q_3(x,t), \quad 0\le x\le 1, 0\le t<\infty,\\ u(x,y,0)=&q_4(x,y),\, \frac{\partial u(x,y,t)}{\partial t}|_{t=0}=q_5(x,y), \quad 0\le x\le 1, 0\le y\le 1,\end{aligned} \tag{38}$$

where $q_0(y,t),\ q_1(y,t),\ q_2(x,t),\ q_3(x,t),\ q_4(x,y)$, and $q_5(x,y)$ are given functions.

Presently, we assume the accompanying transformation

$$\begin{aligned} V(x,y,t) = v(x,y,t) + a_0(y,t) + xa_1(y,t) + x(x-1)(b_0(x,t) + yb_1(x,t)) \\ + x(x-1)y(y-1)(c_0(x,y) + tc_1(x,y)), \end{aligned} \tag{39}$$

where

$$\begin{aligned}
a_0(y,t) &= -q_0(y,t), \quad a_1(y,t) = q_0(y,t) - q_1(y,t),\\
b_0(x,t) &= \frac{(1-x)q_0(0,t) - q_2(x,t) + xq_1(0,t)}{x(x-1)},\\
b_1(x,t) &= \frac{(q_2(x,t) - q_3(x,t)) + (1-x)(q_0(1,t) - q_0(0,t)) + x(q_1(1,t) - q_1(0,t))}{x(x-1)},\\
c_0(x,y) &= \frac{(q_2(x,0) + q_0(y,0) - q_4(x,y)) + x(q_1(y,0) - q_1(0,0) - q_0(y,0)) - q_0(0,0)(1-x)}{xy(x-1)(y-1)}\\
&\quad + \frac{y(q_3(x,0) - q_2(x,0)) + y(1-x)(q_0(0,0) - q_0(1,0)) + xy(q_1(0,0) - q_1(1,0))}{xy(x-1)(y-1)},\\
c_1(x,y) &= \left[\frac{(\frac{\partial q_0(y,t)}{\partial t} - q_5(x,y) - \frac{\partial q_0(0,t)}{\partial t} + \frac{\partial q_2(x,t)}{\partial t}) + x(\frac{\partial q_1(y,t)}{\partial t} - \frac{\partial q_0(y,t)}{\partial t} + \frac{\partial q_0(0,t)}{\partial t} - \frac{\partial q_1(0,t)}{\partial t})}{xy(x-1)(y-1)}\right]_{t=0}\\
&\quad + \left[\frac{y(\frac{\partial q_3(x,t)}{\partial t} - \frac{\partial q_2(x,t)}{\partial t}) + y(1-x)(\frac{\partial q_0(0,t)}{\partial t} - \frac{\partial q_0(1,t)}{\partial t}) + xy(\frac{\partial q_1(0,t)}{\partial t} - \frac{\partial q_1(1,t)}{\partial t})}{xy(x-1)(y-1)}\right]_{t=0}.
\end{aligned}$$

The transformation (39) turns the nonhomogeneous initial-boundary conditions (38) into the homogeneous initial-boundary conditions

$$V(0,y,t) = V(1,y,t) = V(x,0,t) = V(x,1,t) = V(x,y,0) = \frac{\partial V(x,y,t)}{\partial t}|_{t=0} = 0, \tag{40}$$

Subsequently it suffices to solve the following 2D linear HTTE:

$$\begin{aligned} \frac{\partial^2 V(x,y,t)}{\partial t^2} + 2\rho(x,y,t)\frac{\partial V(x,y,t)}{\partial t} + \varrho^2(x,y,t)V(x,y,t) = \sigma(x,y,t)(\frac{\partial^2 V(x,y,t)}{\partial x^2} \\ + \frac{\partial^2 V(x,y,t)}{\partial y^2}) + \mathcal{G}(x,y,t),\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le t < \infty, \end{aligned} \tag{41}$$

and 2D nonlinear HTTE:

$$\begin{aligned} \frac{\partial^2 V(x,y,t)}{\partial t^2} + 2\rho(x,y,t)\frac{\partial V(x,y,t)}{\partial t} - \sigma(x,y,t)(\frac{\partial^2 V(x,y,t)}{\partial x^2} + \frac{\partial^2 V(x,y,t)}{\partial y^2}) \\ = \mathcal{G}(x,y,t,V),\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le t < \infty. \end{aligned} \tag{42}$$

5. NUMERICAL RESULTS

In this Section, we consider some numerical examples to offer the accuracy of the proposed method. The distinction between the measured value of approximate solution and its actual value (absolute error), is given by

$$E(x,t) = |v(x,t) - \widetilde{v}(x,t)|, \tag{43}$$

$$E(x,y,t) = |v(x,y,t) - \widetilde{v}(x,y,t)|, \tag{44}$$

where $v(x,t)$ and $\widetilde{v}(x,t)$ are the exact solution and the numerical solution at the point (x,t), respectively. Also $v(x,y,t)$ and $\widetilde{v}(x,y,t)$ are the exact solution and the numerical solution at the point (x,y,t), respectively. Moreover, the maximum absolute errors (MAEs) is given by

$$\text{MAEs} = \text{Max}\{E(x,t) : \forall (x,t) \in [0,1] \times [0,\infty)\} = L^{\infty}. \tag{45}$$

$$\text{MAEs} = \text{Max}\{E(x,y,t) : \forall (x,y,t) \in [0,1] \times [0,1] \times [0,\infty)\} = L^{\infty}. \tag{46}$$

5.1. TEST PROBLEM 1

We consider the accompanying 1D telegraph equation (8) with constant coefficients where $\rho(x,t) = \frac{1}{2}, \varrho^2(x,t) = 1$ and $\sigma(x,t) = 1$ in the domain $0 \le x \le 1$, with initial and boundary conditions,

$$v(0,t) = v(1,t) = 0,\ 0 \le x \le 1,$$
$$v(x,0) = \frac{\partial v(x,t)}{\partial t}|_{t=0} = 0,\ t \ge 0.$$

The exact solution is given by
$v(x,t) = t^2(x - x^2)e^{-t}$ and $\chi(x,t) = e^{-t}\left[t^2\left(-x^2 + x + 2\right) + 2t(x-1)x - 2(x-1)x\right]$.

The absolute error is shown in Table 1 with various choice of x, t, N, and M. Table 2 shows the L^{∞} error at different values of t and $N = M = 10$. Figure 1 shows the exact solution and the numerical solutions of Problem 5.1 at $N = M = 10$.
In Figs. 2 and 3, we depicted the exact $v(x,t)$ and numerical $\widetilde{v}(x,t)$ solutions for Problem 5.1, where $N = M = 10$ at four different values of t and the exact $v(x,t)$ and numerical $\widetilde{v}(x,t)$ solutions for Problem 5.1, where $N = M = 10$ at four different values of x, respectively.

5.2. TEST PROBLEM 2

In this example, the 1D telegraph equation (8) is considered with variable coefficients $\rho(x,t) = 2e^{(x+t)}, \varrho^2(x,t) = \sin^2(x+t)$ and $\sigma(x,t) = 1 + x^2$ in the domain

Table 1

Absolute error with various choice of x, t, N, and M for Problem 5.1.

(x,t)	$N=M=4$	$N=M=6$	$N=M=8$	$N=M=10$
$(0.1,0.1)$	1.80×10^{-4}	1.45×10^{-5}	1.13×10^{-5}	3.71×10^{-6}
$(0.2,0.2)$	1.08×10^{-3}	5.47×10^{-5}	1.99×10^{-5}	1.02×10^{-5}
$(0.3,0.3)$	2.68×10^{-3}	3.46×10^{-4}	2.64×10^{-5}	9.61×10^{-6}
$(0.4,0.4)$	4.50×10^{-3}	6.93×10^{-4}	1.11×10^{-4}	3.39×10^{-6}
$(0.5,0.5)$	5.94×10^{-3}	7.14×10^{-4}	1.74×10^{-4}	4.96×10^{-6}
$(0.6,0.6)$	6.49×10^{-3}	1.20×10^{-4}	1.67×10^{-4}	1.67×10^{-5}
$(0.7,0.7)$	5.89×10^{-3}	1.01×10^{-3}	9.33×10^{-5}	3.31×10^{-5}
$(0.8,0.8)$	4.21×10^{-3}	2.12×10^{-3}	1.57×10^{-6}	4.77×10^{-5}
$(0.9,0.9)$	1.93×10^{-3}	2.19×10^{-3}	4.40×10^{-5}	4.33×10^{-5}

Table 2

L^{∞}-error with various choice of t and $N=M=10$ for Problem 5.1.

t	L^{∞}-error
1	8.45×10^{-5}
2	3.60×10^{-4}
3	1.69×10^{-3}
4	5.57×10^{-2}
5	1.09×10^{-1}

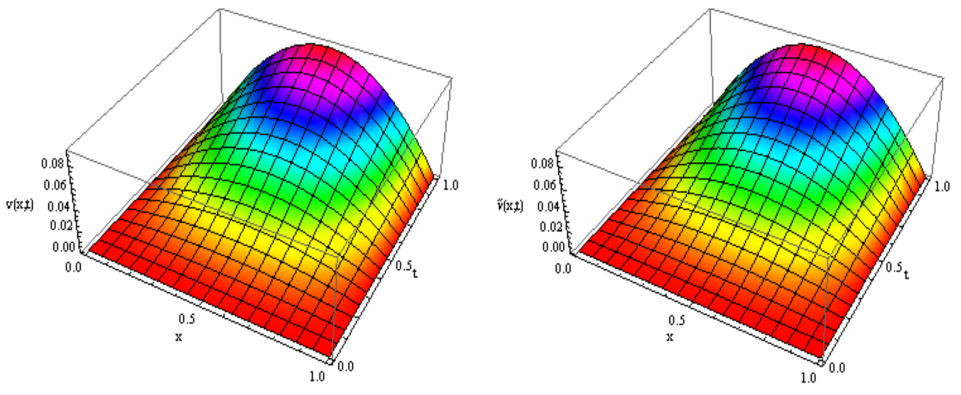

Fig. 1 – The exact $v(x,t)$ and numerical $\widetilde{v}(x,t)$ solutions for Problem 5.1 where $N=M=10$.

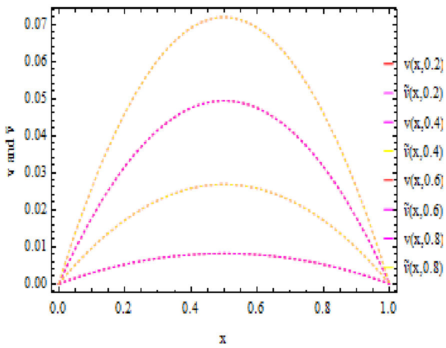

Fig. 2 – The curves-graph of exact $v(x,t)$ and numerical $\widetilde{v}(x,t)$ solutions for Problem 5.1 where $N = M = 10$ at four different values of t.

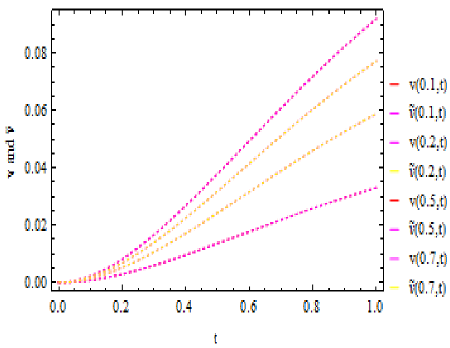

Fig. 3 – The curves-graph of exact $v(x,t)$ and numerical $\widetilde{v}(x,t)$ solutions for Problem 5.1 where $N = M = 10$ at four different values of x.

Table 3

Absolute error using B-LC method for Problem 5.2 with $N = M = 10$.

x	t	E	x	t	E	x	t	E
0.1	0.1	2.35×10^{-6}	0.1	0.5	7.92×10^{-8}	0.1	1	1.34×10^{-5}
0.2		4.52×10^{-6}	0.2		9.21×10^{-8}	0.2		2.54×10^{-5}
0.3		6.34×10^{-6}	0.3		2.49×10^{-8}	0.3		3.47×10^{-5}
0.4		7.64×10^{-6}	0.4		3.14×10^{-7}	0.4		4.01×10^{-5}
0.5		8.27×10^{-6}	0.5		7.76×10^{-7}	0.5		4.08×10^{-5}
0.6		8.13×10^{-6}	0.6		1.34×10^{-6}	0.6		3.67×10^{-5}
0.7		7.18×10^{-6}	0.7		1.88×10^{-6}	0.7		2.83×10^{-5}
0.8		5.41×10^{-6}	0.8		2.13×10^{-6}	0.8		1.72×10^{-5}
0.9		2.94×10^{-6}	0.9		1.70×10^{-6}	0.9		6.16×10^{-6}

$0 \le x \le 1$, with taking after initial and boundary conditions

$$\begin{aligned} &v(0,t) = 0,\ v(1,t) = e^{-2t}\sinh(1),\ t \ge 0,\\ &v(x,0) = \sinh x,\ \frac{\partial v(x,t)}{\partial t}|_{t=0} = -\sinh x,\ 0 \le x \le 1. \end{aligned}$$

The exact solution is given by [34]
$v(x,t) = e^{(-2t)}\sinh(x)$ and $\chi(x,t) = (3 - x^2 + \sin^2(x+t) - 4e^{(x+t)})e^{(-2t)}\sinh(x)$.

Table 3 lists the results obtained by the B-LC method in terms of absolute errors at $N = M = 10$ for $t = 0.1, 0.5$, and 1.0 and some values of x in the finite interval $[0,1]$.

5.3. TEST PROBLEM 3

We consider the accompanying 1D nonlinear hyperbolic telegraph-type equation with variable coefficients where

$$\frac{\partial^2 v(x,t)}{\partial t^2} + x^2\frac{\partial v(x,t)}{\partial t} + v(x,t) - (x+t)\frac{\partial^2 v(x,t)}{\partial x^2} = \chi(x,t,v), 0 \le x \le 1, 0 \le t < \infty, \tag{47}$$

with the initial and boundary conditions,

$$\begin{aligned} &v(0,t) = v(1,t) = 0,\ 0 \le x \le 1,\\ &v(x,0) = \frac{\partial v(x,t)}{\partial t}|_{t=0} = 0,\ t \ge 0. \end{aligned} \tag{48}$$

The exact solution is given by

Table 4

Absolute error with various choice of x, t, N, and M for Problem 5.3.

(x,t)	$N=M=3$	$N=M=5$	$N=M=7$	$N=M=9$
$(0.1,0.1)$	9.35×10^{-4}	4.19×10^{-4}	1.50×10^{-4}	2.23×10^{-4}
$(0.2,0.2)$	5.56×10^{-3}	1.72×10^{-3}	7.46×10^{-4}	9.10×10^{-4}
$(0.3,0.3)$	1.31×10^{-2}	2.71×10^{-3}	1.38×10^{-3}	1.42×10^{-3}
$(0.4,0.4)$	1.94×10^{-2}	2.71×10^{-3}	1.39×10^{-3}	1.30×10^{-3}
$(0.5,0.5)$	1.89×10^{-2}	2.15×10^{-3}	4.86×10^{-4}	5.89×10^{-4}
$(0.6,0.6)$	6.97×10^{-3}	1.86×10^{-3}	8.24×10^{-4}	3.02×10^{-4}
$(0.7,0.7)$	1.69×10^{-2}	2.30×10^{-3}	1.37×10^{-3}	8.66×10^{-4}
$(0.8,0.8)$	4.45×10^{-2}	3.02×10^{-3}	2.23×10^{-4}	8.51×10^{-4}
$(0.9,0.9)$	5.33×10^{-2}	2.78×10^{-3}	1.74×10^{-3}	4.20×10^{-4}

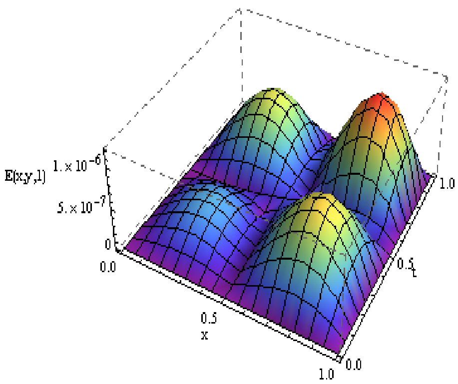

Fig. 4 – The absolute error for Problem 5.4 where $N=M=8$.

$v(x,t) = t^2(x-x^2)\sin(t)e^{-2x}$ and

$$\begin{aligned}\chi(x,t,v) = &-e^{-4x}x\{\sin(t)(2e^{2x}(t^3(-2(x-3)x-3)+t(x-1)x^2+x-1)\\ &-t^4(x-1)^2x\sin(t))+te^{2x}(x-1)(tx^2+4)\cos(t)\}-[v(x,t)]^2.\end{aligned}$$

The maximum absolute errors of $v(x,t)$ related to (47)-(48) are introduced in Table 4 using the B-LC method with four choices of N and M.

5.4. TEST PROBLEM 4

In this problem, the telegraph equation (25) is considered with variable coefficients $\rho(x,y,t) = x^2+y^2+1$, $\varrho^2(x,y,t) = \frac{1}{x^2+t^2}$ and $\sigma(x,y,t) = x^2+y^2+t^2$ in

Table 5

Absolute value of the error $E(x,y,1)$ with various choice of x,t,N, and M for Problem 5.4.

(x,y)	$N=M=4$	$N=M=6$	$N=M=8$
$(0.1,0.1)$	1.65×10^{-6}	1.24×10^{-7}	9.68×10^{-8}
$(0.2,0.2)$	5.53×10^{-6}	3.09×10^{-7}	3.10×10^{-7}
$(0.3,0.3)$	8.47×10^{-6}	1.87×10^{-8}	3.68×10^{-7}
$(0.4,0.4)$	7.21×10^{-6}	1.08×10^{-6}	1.17×10^{-7}
$(0.5,0.5)$	9.22×10^{-7}	2.05×10^{-6}	1.58×10^{-7}
$(0.6,0.6)$	7.76×10^{-6}	1.43×10^{-6}	4.33×10^{-8}
$(0.7,0.7)$	1.38×10^{-5}	1.24×10^{-6}	7.53×10^{-7}
$(0.8,0.8)$	1.28×10^{-5}	3.87×10^{-6}	1.20×10^{-6}
$(0.9,0.9)$	5.53×10^{-6}	2.97×10^{-6}	6.87×10^{-7}

the domain $0\le x\le 1$, with initial and boundary conditions

$$\begin{aligned}
&v(0,y,t)=0,\ v(1,y,t)=\cos(t)\sin(1)\sin(y),\ 0\le y\le 1,\ 0\le t\le 1,\\
&v(x,0,t)=0,\ v(x,1,t)=\cos(t)\sin(x)\sin(1),\ 0\le x\le 1,\ 0\le t\le 1,\\
&v(x,y,0)=\sin(x)\sin(y),\ \frac{\partial v(x,y,t)}{\partial t}|_{t=0}=0,\ 0\le x\le 1,\ 0\le y\le 1.
\end{aligned}$$

The exact solution is given by $v(x,y,t)=\cos(t)\sin(x)\sin(y)$,
and
$\chi(x,y,t)=\sin x\sin y\left[\left(\frac{1}{t^2+x^2}+2t^2+2x^2+2y^2-1\right)\cos t-2\left(x^2+y^2+1\right)\sin t\right].$

The absolute values of the error $E(x,y,1)$ with various choices of x,t,N, and M for Problem 5.4 are provided in Table 5. Moreover, the absolute error is depicted in Fig. 4.

5.5. TEST PROBLEM 5

For this Problem, the nonlinear telegraph equation (34) is considered with variable coefficients $\rho(x,y,t)=xyt$ and $\sigma(x,y,t)=x^2+y^2t$ in the domain $0\le x\le 1$, and initial and boundary conditions

$$\begin{aligned}
&v(0,y,t)=v(1,y,t)=0,\ 0\le y\le 1,\ t\ge 0,\\
&v(x,0,t)=v(x,1,t)=0,\ 0\le x\le 1,\ t\ge 0,\\
&v(x,y,0)=\frac{\partial v(x,y,t)}{\partial t}|_{t=0}=0,\ 0\le x\le 1,\ 0\le y\le 1.
\end{aligned}$$

The exact solution is given by $u(x,y,t)=xyt^2(1-x)(1-y)e^{x+y-t}$,

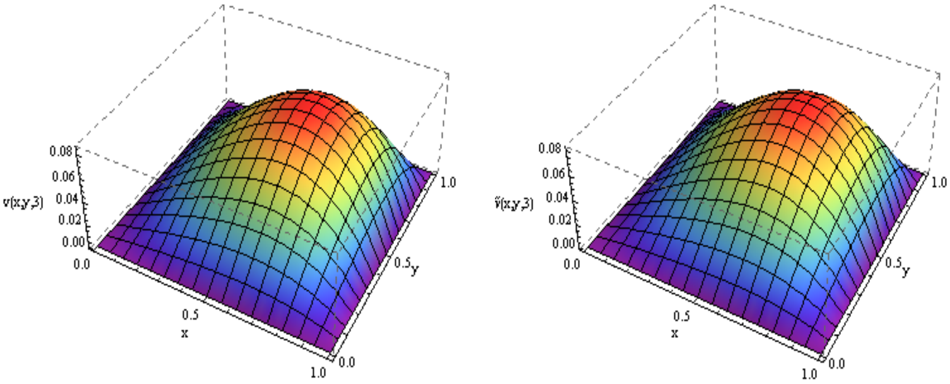

Fig. 5 – The exact $v(x, y, 3)$ and numerical $\widetilde{v}(x, y, 3)$ solutions for Problem 5.5 where $N = M = 6$.

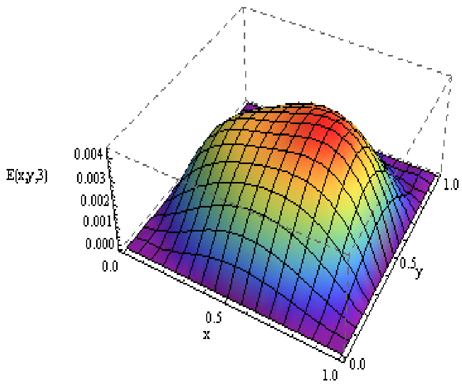

Fig. 6 – The absolute error for Problem 5.5 where $N = M = 6$.

Table 6

Absolute value of the error $E(x,y,1)$ with various choice of x,t,N, and M for Problem 5.5.

(x,y)	$N=M=2$	$N=M=4$	$N=M=6$
$(0.1,0.1)$	3.00×10^{-3}	6.03×10^{-4}	5.38×10^{-5}
$(0.2,0.2)$	1.20×10^{-2}	1.95×10^{-3}	1.24×10^{-5}
$(0.3,0.3)$	2.61×10^{-2}	2.90×10^{-3}	5.15×10^{-5}
$(0.4,0.4)$	4.26×10^{-2}	2.38×10^{-3}	7.16×10^{-5}
$(0.5,0.5)$	5.76×10^{-2}	3.29×10^{-4}	1.56×10^{-4}
$(0.6,0.6)$	6.58×10^{-2}	2.14×10^{-3}	3.68×10^{-4}
$(0.7,0.7)$	6.23×10^{-2}	3.37×10^{-3}	5.60×10^{-4}
$(0.8,0.8)$	4.46×10^{-2}	2.52×10^{-3}	5.14×10^{-4}
$(0.9,0.9)$	1.73×10^{-2}	6.90×10^{-4}	2.21×10^{-4}

and

$$\begin{aligned}\chi(x,y,t,v) = &-(v(x,y,t))^2 + xye^{-2t+x+y}(t^4(x-1)^2x(y-1)^2ye^{x+y}\\ &-e^t(2t^3y(x^2(y-1)+x(y^2+1)+(y-3)y)+t^2(x(-4(x-1)y^2\\ &+2x(x+3)y+2(x-3)x-5y+1)+y-1)+4t(x-1)(y-1)\\ &+2(x(-y)+x+y-1))).\end{aligned}$$

Table 6 shows the absolute values of the error $E(x,y,1)$ with various choices of x,t,N, and M for Problem 5.5. The graph of exact $v(x,y,3)$ and numerical $\widetilde{v}(x,y,3)$ solutions for Problem 5.5, where $N=M=6$ are shown in Fig. 5. Moreover, the absolute errors are depicted in Fig. 6.

6. CONCLUSION

Bernoulli-Laguerre collocation method was used to solve linear and nonlinear one- and two-dimensional hyperbolic telegraph-type equations with variable coefficients. For this purpose we utilized Bernoulli-Laguerre-Gauss nodes to reduce the considered hyperbolic telegraph-type equations to the solution of a matrix equation. In this way we proved the utility of our approach *versus* other analytical or numerical approaches.

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