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# ON ELEMENTARY DERIVATION OF GREEN'S FUNCTION OF WAVE EQUATION

ALEXEY E. RASTEGIN

Department of Theoretical Physics,
Irkutsk State University,
Gagarin Bv., 20, Irkutsk, 664003, Russia
Email: *alexrastegin@mail.ru*

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*Abstract.* Green's functions are typically derived with the use of mathematics with which undergraduate students are not familiar. Educational programs may be organized so that the wave equation is learned before the course of mathematical physics. Meanwhile, the technique of using Green's functions can be introduced at undergraduate level. In this paper, a simple approach to derivation of Green's function of the wave equation is proposed. It demands no more than usual reasons typical for studies of Poisson's equation. One starts with the divergence theorem and the Laplacian expressed in spherical coordinates. The proposed approach is illustrated with a lot of examples. They are connected with Huygens' principle and Euler's analysis of the vibrating string problem.

*Key words*: physics education, problem-based learning, wave phenomena.

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One of important goals of the modern education is to stimulate creative capacities in students by problem-solving approach [1]. The notion of a wave is something familiar to everyone in one form or another. The wave equation is typically used in various models which represent physics in a number of different settings. However, there are rather few working examples for learning the wave equation by undergraduate students. In comparison, problem-solving approach can easily be realized with respect to Poisson's equation. Freshman-level courses of electromagnetism are usually filled with many examples on calculation of the electrostatic field at the given distribution of charge. Although Poisson's equation often remains in a background, it appears naturally in students' understanding.

An opposite situation takes place with the wave equation. The wave equation *per se* is included into freshman-level courses, but its consideration is typically restricted to plane waves. Non-homogeneous problems for the wave equation are almost excluded from consideration at this stage. Such problems are assumed to be examined in later advanced courses, when students will be familiar with powerful mathematical methods. Here, we have come across breaking a continuous process of problem-based learning. In addition, advanced courses of physics are beyond interests of students oriented on engineering disciplines. Such students typically learn physics only in a general course. It is of interest to find an elementary approach to retarded solutions of the non-homogeneous wave equation.

The simplest case arises when the physical field of interest is described by a scalar potential function $\mathit{\Phi}(\mathbf{r},t)$. The non-homogeneous wave equation is written as

$$\boldsymbol{\nabla}^2\mathit{\Phi}-\frac{1}{c^2}\,\frac{\partial^2\mathit{\Phi}}{\partial t^2}=Q(\mathbf{r},t)\,. \tag{1}$$

Here, the function $Q(\mathbf{r},t)$ describes acting source; its physical meaning is determined by the nature of considered processes. Due to linearity of the model (1), it allows superposition solutions. If the source $Q$ is represented as the sum of two others, say $Q_1$ and $Q_2$, and we already know some solutions $\mathit{\Phi}_1$ and $\mathit{\Phi}_2$ for each of them separately, then a solution of (1) is merely obtained as

$$\mathit{\Phi}(\mathbf{r},t)=\mathit{\Phi}_1(\mathbf{r},t)+\mathit{\Phi}_2(\mathbf{r},t)\,. \tag{2}$$

This approach naturally leads to a method for solving non-homogeneous linear models known as the technique of Green's functions [2]. The given source is represented as superpositions of some "elementary" sources, for which solutions have already been found. It is explained in many books intended for readers at different levels. We aim to motivate that the mathematics of Green's functions could be introduced earlier in the physics curriculum.

The method of representation of arbitrary source as a superposition of elementary ones is probably most familiar to those who learn Poisson's equation. It will be instructive to discuss previously just this case. Let us consider the non-homogeneous equation

$$\boldsymbol{\nabla}^2\mathit{\Psi}=\rho(\mathbf{r})\,. \tag{3}$$

Here, $\mathit{\Psi}(\mathbf{r})$ is some abstract potential, whereas $\rho(\mathbf{r})$ describes the distribution of sources. The equation can be solved by means of the integral

$$\mathit{\Psi}(\mathbf{r})=\int G(\mathbf{r},\mathbf{R})\,\rho(\mathbf{R})\,\mathrm{d}V_R\,. \tag{4}$$

Here, $G(\mathbf{r},\mathbf{R})$ denotes Green's function with the standard expression,

$$G(\mathbf{r},\mathbf{R})=-\frac{1}{4\pi|\mathbf{r}-\mathbf{R}|}\,. \tag{5}$$

It actually depends only on $|\mathbf{r}-\mathbf{R}|$. In one of memoirs, Poisson explained his approach to the equation for gravitational potential, — it is considered in detail in items 1241–3 of the well-known historical study [3]. Apparently, Poisson was the first who has noted that Laplace's equation for gravitational potential is valid only outside of a solid. The basic points of Poisson's approach are very close to the modern formulation (4).

Today, students often meet similar reasons in the context of electrostatics. These questions concern Gauss' law, although first courses of electricity typically avoid to use explicitly the delta function. The point charge $q_0$ at the origin gives the charge density $q_0\,\delta(\mathbf{r}) = q_0\,\delta(x)\,\delta(y)\,\delta(z)$. It is worthy to remember this fact, when Poisson's equation is learned in the context of electrostatics. Although roots of the delta function can be traced back to the early 19th century, only Heaviside and then Dirac were those who gave it an independent role [4]. Dirac's delta function provides a clear technique for dealing with point sources. Basic skills can quite be obtained by beginners, say, on the base of chapter 16 of the textbook [5]. An informal definition is that $\delta(x) = 0$ for all $x \neq 0$ and, for any usual function $h(x)$,

$$\int \delta(x)\,h(x)\,\mathrm{d}x = h(0)\,, \tag{6}$$

whenever the point $x = 0$ lies between the limits of integration. It is useful to recall that the delta function is even and $\delta(\alpha x) = |\alpha|^{-1}\,\delta(x)$ for a non-zero constant $\alpha$.

In general courses of physics, undergraduate students have arrived at the formulas (4) and (5) usually in electrostatics. Gauss' law and the divergence theorem provide a ground for these. Similar ideas are useful in derivation of Green's function of the wave equation. To do so, we take a time-varying point source located at the coordinate origin:

$$Q(\mathbf{r},t) = \delta(\mathbf{r})\,q(t)\,. \tag{7}$$

The problem is to solve (1) with the source (7). Due to spherical symmetry of the problem, solutions should be of the form $\Phi(r,t)$, where $r = |\mathbf{r}|$. The radial part of the Laplacian can be expressed as

$$\boldsymbol{\nabla}^2\Phi(r,t) = \frac{1}{r}\,\frac{\partial^2}{\partial r^2}\,r\,\Phi(r,t)\,. \tag{8}$$

For $r \neq 0$, an auxiliary function $f(r,t) = r\,\Phi(r,t)$ obeys homogeneous wave equation with one spatial variable:

$$\frac{\partial^2 f}{\partial r^2} - \frac{1}{c^2}\,\frac{\partial^2 f}{\partial t^2} = 0\,. \tag{9}$$

Due to d'Alembert's formula, $f(r,t)$ is the sum of two terms that depend on $r \pm ct$. It is useful to write the solution as $f(r,t) = u(t - r/c) + v(t + r/c)$. The signal propagates as a spherically diverging wave, whence we take $v \equiv 0$. We should now determine one-variable function $u$ according to the equation (1) with the source (7).

As is seen from experience with Gauss' law, a good way is to integrate the equation of interest over a small volume containing the origin. The first part of the raised equation will be transformed into the surface integral, similarly to the case of

Poisson's equation. Substituting $\mathit{\Phi}(r,t)=u(t-r/c)/r$ and integrating over a ball leads to the condition

$$\int_{r\leq a}\mathrm{d}V\left(\boldsymbol{\nabla}^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\frac{u(t-r/c)}{4\pi r}=\frac{q(t)}{4\pi}\,, \tag{10}$$

where $a$ is radius of the ball. To simplify subsequent formulas, we have divided our equation by the total solid angle $4\pi$. The left-hand side of (10) appeared as the sum of two terms. The first term is transformed into the surface integral over the sphere $r=a$ and equal to

$$a^2\frac{\partial}{\partial a}\frac{u(t-a/c)}{a}=-\frac{a}{c}\,u'(t-a/c)-u(t-a/c)\,. \tag{11}$$

To the second term of the left-hand side of (10) we apply integration by parts, whence

$$\begin{aligned}-\frac{1}{c^2}\int_0^a u''(t-r/c)\,r\,\mathrm{d}r&=\frac{r}{c}\,u'(t-r/c)\Big|_0^a-\frac{1}{c}\int_0^a u'(t-r/c)\,\mathrm{d}r\\&=\frac{a}{c}\,u'(t-a/c)+u(t-a/c)-u(t)\,.\end{aligned} \tag{12}$$

Combining (11) and (12) with (10) gives $-u(t)=q(t)/(4\pi)$, whence the answer follows.

To sum up, we see the following. The equation (1) with the source $Q(\mathbf{r},t)=\delta(\mathbf{r})\,q(t)$ is satisfied by

$$\mathit{\Phi}(r,t)=-\frac{q(t-r/c)}{4\pi r}\ . \tag{13}$$

It is useful to compare this with the answer for $\boldsymbol{\nabla}^2\mathit{\Psi}=q_0\,\delta(\mathbf{r})$, namely

$$\mathit{\Psi}(r)=-\frac{q_0}{4\pi r}\ . \tag{14}$$

The reader familiar with electrodynamics can consider the results with mentioning distinctions and similarities of the electrodynamic and electrostatic cases. The former differs from the latter by time-varying source taken at the retarded time. Of course, this question is much deeper than it can be seen from comparing scalar equations and their solutions. Substituting $q(t)=\delta(t)$ finally leads to the function

$$g(r,t)=-\frac{\delta(t-r/c)}{4\pi r}\ . \tag{15}$$

The wave operator acting on (15) gives $\delta(\mathbf{r})\,\delta(t)$, whence $g(r,t)$ is a fundamental solution of the operator. The right-hand side of (15) is clearly zero for $t<0$, so that the Heaviside function $H(t)$ can be added explicitly as a factor. Replacing $t$ with $t-\tau$ and $r$ with $|\mathbf{r}-\mathbf{R}|$ leads to the basic Green's function in infinite space with no boundaries. In other domains, solutions of homogeneous wave equation should be added in order to satisfy imposed boundary conditions. To solve the non-homogeneous equation (1), we will calculate the convolution

$$\Phi(\mathbf{r},t) = \int \mathrm{d}V_R \int \mathrm{d}\tau\, g\big(|\mathbf{r}-\mathbf{R}|, t-\tau\big)\, Q(\mathbf{R},\tau)\,. \tag{16}$$

This expression for solutions of the wave equation is similar to the formula (4). The latter should be well known to any student who had learned Poisson's equation.

The standard derivation of (15) can be found in many textbooks. It is an example of more general approach with the Fourier transform followed by contour integration. The latter demands some rule of handling the singularities. The question is resolved from physical considerations as a kind of radiation conditions. This technique is an important mathematical tool of theoretical physics. On the other hand, such an approach is rather ill-timed for undergraduate students. The elementary derivation described above may be used in learning of non-homogeneous problems by beginners. In addition, the formula (13) gives an answer with arbitrary time dependence $q(t)$ of the point source. In any case, the existence of several approaches may give an additional insight. The following example can be recommended to students for self-studying.

*Exercise 1.* Find solution of the equation

$$\frac{\partial^2\phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \delta(x)\, q(t)\,,$$

corresponding to a diverging wave. Substituting $q(t) = \delta(t)$, obtain a fundamental solution of the wave operator in one dimension.

We have considered a solution of the non-homogeneous wave equation with the source $\delta(\mathbf{r})\, q(t)$. It is instructive to examine an instantaneous source with arbitrary spatial dependence. So, we ask a solution of the equation

$$\boldsymbol{\nabla}^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = p(\mathbf{r})\,\delta(t)\,. \tag{17}$$

Due to the general approach, we now substitute $Q(\mathbf{R},\tau) = p(\mathbf{R})\,\delta(\tau)$ into the formula (16). A convenient way to deal with the integral is to introduce the vector variable $\boldsymbol{\xi} = \mathbf{R}-\mathbf{r}$. Calculations finally lead to the expression

$$\begin{aligned}\Phi(\mathbf{r},t) &= -\frac{H(t)\,c}{4\pi}\int \mathrm{d}\Omega_\xi \int_0^\infty \xi\, p(\mathbf{r}+\boldsymbol{\xi})\,\delta(\xi-ct)\,\mathrm{d}\xi \\ &= -\frac{H(t)\,c^2 t}{4\pi}\int p(\mathbf{r}+ct\mathbf{e}_\xi)\,\mathrm{d}\Omega_\xi\,,\end{aligned} \tag{18}$$

where the unit vector $\mathbf{e}_\xi = \boldsymbol{\xi}/\xi$. The value of potential at time $t$ in the observation point is obtained by averaging of the spatial source distribution over the sphere of radius $ct$ with center at this point. This results concurs with the treatment of wave propagation on the base of Huygens' principle.

Let us consider a concrete example of spherically symmetric distribution. We take it in the form of Gaussian function

$$p_1(r) = \frac{B}{\pi^{3/2}a^3}\,\exp(-r^2/a^2)\,, \tag{19}$$

where $B = \mathrm{const}$. Substituting (19) into (18), one finally obtains

$$\Phi_1(\mathbf{r},t) = \frac{H(t)\,ca^2}{4r}\Big(-p_1(r-ct) + p_1(r+ct)\Big)\,. \tag{20}$$

The size $a$ characterizes a spatial scale of the source. In the far field, where $r \gg a$, the first term in the right-hand side of (20) is quite sufficient. Indeed, the second one is exponentially decayed therein. The near-field description require both the terms that together provide a finite value for $r \to 0$. In a point of the far zone, we will observe an impulse proportional to $p_1(r-ct)$.

*Exercise 2.* Consider (20) in the limit $a \to 0$, when the right-hand side of (19) becomes $B\,\delta(\mathbf{r})$. Here, we recall one of representations of the delta function, namely (cf. the equation (16) of the paper [4])

$$\frac{1}{a\sqrt{\pi}}\,\exp(-x^2/a^2) \underset{a\to 0}{\longrightarrow} \delta(x)\,. \tag{21}$$

Note that for $a \to 0$, the far field occupies the whole space, except for the point $\mathbf{r} = \mathbf{0}$. To check a self-consistency, the limiting case of the function (20) should be compared with (15) multiplied by $B$.

The result (18) can be used in analysis of the problem with initial conditions as non-homogeneous. Let us consider the following initial conditions:

$$\Phi(\mathbf{r},t)\big|_{t=0} = 0\,, \qquad \frac{\partial\Phi(\mathbf{r},t)}{\partial t}\bigg|_{t=0} = w(\mathbf{r})\,. \tag{22}$$

If we ask the solution of homogeneous wave equation with the initial conditions (22), one should determine $\Phi(\mathbf{r},t)$ for $t \in (0;+\infty)$. Instead of this formulation, the question can be reformulated as non-homogeneous with an instantaneous source acting at the moment $t = 0$. On the time axis $t \in (-\infty;+\infty)$, one considers

$$\frac{\partial^2\Phi}{\partial t^2} - c^2\,\boldsymbol{\nabla}^2\Phi = w(\mathbf{r})\,\delta(t)\,, \tag{23}$$

and assumes that $\Phi(\mathbf{r},t) = 0$ for all $t < 0$. To see a consistence with the conditions (22), we integrate (23) over a small interval of the point $t = 0$ and further take this interval to tend to 0. In the left-hand side of (23), the second term does not contribute, whereas the first one leads to an instantaneous jump of $\partial\Phi/\partial t$ from 0 just to $w(\mathbf{r})$. Comparing (23) with (17), we substitute $p(\mathbf{r}) = -c^{-2}w(\mathbf{r})$ into (18). Of course, the above approach can also be applied to the case of non-zero initial value of $\Phi$. We avoid this point, as it deals with the derivative of the delta function.

When the conditions (22) are spherically symmetrical, *i.e.*, depend on $r$ solely, we can further simplify the resulting integral. This case deserves an explicit formulation, since the answer can also be obtained from Euler's analysis of the vibrating string problem. One basic use of the wave equation in one dimension is to model small vibrations of a stretched string. Let $y(x,t)$ measure the string displacement in position $x$ and time $t$. Euler claimed that the final expression for $y(x,t)$ should be deduced from initial conditions [6]. If $y_0(x)$ and $y_1(x)$ are the initial position and velocity of the string, then Euler's solution reads

$$y(x,t) = \frac{1}{2}\Big(y_0(x-ct)+y_0(x+ct)\Big)+\frac{1}{2c}\int_{x-ct}^{x+ct} y_1(s)\,\mathrm{d}s\,. \tag{24}$$

To use this formula, one will extend the curves $y_0(x)$ and $y_1(x)$ from the range $0 \le x \le L$ along the real axis. Euler proclaimed that extending should be done with odd periodicity. The solution (24) was a part of principal debate between d'Alembert, Euler, and Daniel Bernoulli [6]. We only notice that (24) could be used in solving (9) with initial conditions. Recall that $f(r,t) = r\Phi(r,t)$ here. For the initial conditions (22) with a dependence only on $r$, we now write

$$f(r,t)\big|_{t=0} = 0\,, \qquad \frac{\partial f(r,t)}{\partial t}\bigg|_{t=0} = r\,w(r)\,. \tag{25}$$

Using Euler's solution, we obtain the answer for $f(r,t)$, whence

$$\Phi(r,t) = \frac{1}{2cr}\int_{r-ct}^{r+ct} s\,w(s)\,\mathrm{d}s\,. \tag{26}$$

Due to Euler, the integrand initially prescribed for positive $s$ should be treated here as an odd function. Of course, we will use (26) only for $t>0$.

Let us obtain the same solution from (18). Substituting $p(\mathbf{r}) = -c^{-2}w(r)$, we have

$$\frac{H(t)\,t}{4\pi}\int w\big(|\mathbf{r}+ct\mathbf{e}_\xi|\big)\,\mathrm{d}\Omega_\xi = \frac{H(t)\,t}{2}\int_{-1}^{+1} w\big(\sqrt{r^2+2rct\mu+(ct)^2}\,\big)\,\mathrm{d}\mu\,, \tag{27}$$

where $\mu = \cos\theta$ and $\theta$ is the angle between $\mathbf{r}$ and $\mathbf{e}_\xi$. Introducing the new variable $s = \sqrt{r^2+2rct\mu+(ct)^2}$, for which $\mathrm{d}\mu = s\,\mathrm{d}s/(rct)$, we finally get the answer

$$\Phi(r,t) = \frac{H(t)}{2cr}\int_{|r-ct|}^{r+ct} s\,w(s)\,\mathrm{d}s\,, \tag{28}$$

which depends on $r$. Due to the Heaviside function, the result vanishes for $t<0$. For $t>0$, the integral (28) almost coincides with (26). A form of the lower limit of integration is the only distinction. Recall now that the integrand should be taken as odd in the solution (26). Hence, both the integrals actually give the same value, even if $r - ct < 0$. This example illustrates power of the technique of using Green's function.

*Exercise 3.* Using Euler's solution (24), consider a string plucked in the middle with zero initial velocity. Explain and visualize changes of the initial isosceles-triangle form with time.

I have proposed simple and concise derivation of Green's function of the wave equation. Related reasoning is similar to that is used in initial learning of electrostatics. In comparison, novelty is connected with a dependence on time. The approach demands only knowledge of the divergence theorem and the Laplacian. It is assumed to be useful for undergraduate courses, when audience is not familiar with advanced mathematical methods. Keeping Green's function, students will be better prepared for a practice with wave problems. The suggested approach can also be learned by those who are interested in engineering disciplines and learn physics only in a general course. Some examples appropriate for the classroom were presented. According to features of his/her own audience, a teacher may increase a number of working examples for students.

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