Photon dynamics in the gravitational field in 4D and its 5D extension

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Extended space model (ESM) is a generalization of the special theory of relativity at a 5-dimensional space, and more specifically at (1 + 4)-dimensional space. Rotations in extended space correspond to the motion of a particle in gravity field in the embedded four-dimensional space-time. The possibility of a transition from the components of the 5-momentum of a particle in extended flat space to the components of a 4-momentum in an arbitrary 4-dimensional space by means of rotations is considered.

Variational principle of the stationary energy integral of photon allows us to determine its dynamics. We consider variation of energy of the light-like particle in the pseudo-Riemann space-time, find Lagrangian, canonical momenta and forces. We study how (TS)-rotation in ESM agrees with photon dynamics in the Schwarzschild field. Equations of the critical curve are obtained by the nonzero energy integral variation in accordance with principles of the calculus of variations in mechanics. This method is compared with the Fermat's principle and geodesics principle. Energy and momentum of the particle transferred to the gravity field is defined. The produced equations are solved for the metrics of Schwarzschild and Goedel. The gravitation mass of the photon is found in central gravity field in the Newtonian limit.

1 Introduction

It is known, that between the mechanical and optical phenomena there is a certain likeness, which historically was exhibited that a set of the optical phenomena managed uniformly well to be described both within the framework of wave, and within the framework of the corpuscular theories. In particular, motion of a beam of light in an inhomogeneous medium in many respects similar to motion of a material particle in a potential field [1]. In the given activity, we shall take advantage of this connection to describe the gravitational phenomena.

The Fermat principle is the basis of geometric optics in media. It is also formulated for

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Riemannian space-time [2, 3]. In [4, 5, 6] it is proposed a variational principle of the stationary energy integral of a light-like particle, which does not lead to violation of the isotropy of light path and agrees with Fermat's principle for stationary gravitational fields. It is also applicable to non-stationary gravitational fields in which the particle motion is free. This approach is the choice of Lagrangian of the particle and the definition of canonical momenta and forces as its partial derivatives with respect to the velocities and coordinates in accordance with Lagrange's mechanics. A correspondence is established between the physical energy and momentum of the particle, determined from non-gravitational interactions, and the contravariant canonical momentum vector.

In [7, 8, 9] it is investigated a generalization of special theory of relativity in a 5-dimensional space $G(1,4)$ with a metric (+ - - - -) having an additional coordinate s. In ESM, in addition to the rotations in plane (TX) relating to the Lorentz transformations, the rotations in planes (TS) and (XS) are considered. In this paper we study how (TS)-rotation agrees with photon dynamics in the Schwarzschild field, which is analyzed using the principle of extreme energy of a light-like particle based on Lagrangian mechanics. The possibility of transition from the 5-momentum components of a particle to an extended space to the components of a 4-momentum in an arbitrary 4-dimensional space by means of a combination of rotations is considered.

2 (TS)-rotation in Extended Space Model

In Minkowski space $M(1,3)$ a 4-vector of energy and momentum

$$\tilde{p} = \left(\frac{E}{c}, p_x, p_y, p_z\right) \tag{1}$$

is associated to each particle [2]. In the extended space $G(1,4)$ [7, 8, 9] it is completed to 5-vector

$$\overline{p} = \left(\frac{E}{c}, p_x, p_y, p_z, mc\right), \tag{2}$$

where m is a rest mass of the particle. In blank space in a fixed reference system there are two types of various objects with zero and nonzero masses. In space $G(1,4)$ to them there corresponds 5-vectors

$$\overline{p}_f = \left(\frac{\hbar\omega}{c}, \frac{\hbar\omega}{c}, 0\right) \tag{3}$$

$$\overline{p}_m = (mc, 0, mc). \tag{4}$$

For simplicity we have recorded these vectors in (1 + 2)-dimensional space. The vector $\overline{p}_f$ describes a photon with energy $\hbar\omega$ and with speed c. The vector $\overline{p}_m$ describes a fixed particle with a rest mass m. Next we will consider the motion of a photon.

At hyperbolic rotations on an angle θ in the plane (TS) the photon vector (4) will be transformed as follows:

$$\left(\frac{\hbar\omega}{c}, \frac{\hbar\omega}{c}, 0\right) \rightarrow \left(\frac{\hbar\omega}{c}\cosh\theta, \frac{\hbar\omega}{c}, \frac{\hbar\omega}{c}\sinh\theta\right) = \left(\frac{\hbar\omega}{c}n, \frac{\hbar\omega}{c}, \frac{\hbar\omega}{c}\sqrt{n^2-1}\right). \tag{5}$$

In ESM this rotation is associated with the photon's motion in a medium in enclosed three-dimension space with refraction index $n > 1$. In such areas the speed of light is reduced. The parameter n relates the speed of light in vacuum c with the speed of light in a medium v as

$$n = \frac{c}{v}. \tag{6}$$

3 Lagrangian, four-momentum and four-force in curved space-

time

In [4, 5, 6] it is proposed a variational principle of the stationary energy integral of photon without violation of Lorentz-invariance. In it the interval in pseudo-Riemann space-time with metrical coefficients $\tilde{g}_{11}$:

$$ds^2 = \tilde{g}_{ij} dx^i dx^j \tag{7}$$

after substitutions

$$\tilde{g}_{11} = \rho^2 g_{11}, \tilde{g}_{1k} = \rho g_{1k}, \tilde{g}_{kq} = g_{kq} \tag{8}$$

is rewritten in form

$$ds^2 = \rho^2 g_{11} dx^{12} + 2\rho g_{1k} dx^1 dx^k + g_{kq} dx^k dx^q. \tag{9}$$

Here, ρ is some quantity, which is assumed to be equal 1. Putting down x^1 as time, coordinates with indexes $k, q = 2,3,4$ as space coordinates and considering ρ as energy of light-like particle with $ds = 0$ we present it as

$$\rho = \left(g_{11} \frac{dx^1}{d\mu}\right)^{-1} \left\{-g_{1k} \frac{dx^k}{d\mu} + \sigma \left[(g_{1k} g_{1q} - g_{11} g_{kq}) \frac{dx^k}{d\mu} \frac{dx^q}{d\mu}\right]^{1/2}\right\}, \tag{10}$$

where σ is ± 1 and μ is affine parameter.

The partial derivatives with respect to coordinates are written as

$$\frac{\partial \rho}{\partial x^\lambda} = -\frac{1}{2u_1 u^1} \frac{\partial g_{ij}}{\partial x^\lambda} u^i u^j, \tag{11}$$

where $u^i = dx^i / d\mu$ is four-velocity vector. The partial derivatives with respect to components of the velocity four-vector are

$$\frac{\partial \rho}{\partial u^\lambda} = -\frac{u_\lambda}{u_1 u^1}. \tag{12}$$

With $g_{11} = 0$ and $g_{1k} \neq 0$ even if for one k the energy takes form

$$\rho = \frac{g_{kq} u^k u^q}{2u_1 u^1}. \tag{13}$$

In this case the partial derivatives of ρ coincide with (11) and (12).

For the free moving a particle lagrangian is taken in form

$$L = -\rho, \tag{14}$$

and conforms to relation [10]:

$$\rho = u^\lambda \frac{\partial L}{\partial u^\lambda} - L. \tag{15}$$

Thus energy ρ is a Hamiltonian of the particle in gravitational field also an integral of the motion. Obtained derivatives give the canonical momenta

$$p_\lambda = \frac{\partial L}{\partial u^\lambda} = \frac{u_\lambda}{u^1 u_1} \tag{16}$$

and forces

$$F_\lambda = \frac{\partial L}{\partial x^\lambda} = \frac{1}{2u^1 u_1} \frac{\partial g_{ij}}{\partial x^\lambda} u^i u^j. \tag{17}$$

Components of the associated vector of the canonical momenta are

$$p^\lambda = \frac{u^\lambda}{u^1 u_1}. \tag{18}$$

Physical energy and momenta of photon with frequency ω in Minkowski space-time with affine parameter $\mu = ct$ form contravariant 4-vector of momenta $\pi^i = (\hbar\omega/c) u^i$. For arbitrary affine parameter it is rewritten as

$$\pi^i = \frac{\hbar\omega}{c}\frac{u^i}{u^1}. \tag{19}$$

And in pseudo-Riemannian space-time similar energy and momenta of the photon will be put in line with the components of the contravariant vector of momenta. A certain fixed value of the photon's frequency ω_0 is given by the corresponding equality $\omega = \omega_0/u_1$. Comparing expressions (18) and (19), we obtain

$$\pi^i = \frac{\hbar\omega_0}{c}p^i. \tag{20}$$

This one provides Lagrangian of the photon $L_{ph} = \hbar\omega_0 L$. The components of vector

$$F^k = g^{k\lambda}F_\lambda \tag{21}$$

associated to (17), with this approach, are proportional to gravity forces:

$$Q^i = \hbar\omega_0 F^i, \tag{22}$$

which acts on the photon. That is, although non-straight motion of particle in space-time according to the general relativity due to its curvature, identified with the gravitational field, we believe that it is caused by the action of forces obtained by considering the movement in the coordinate frame.

4 Equations of Isotropic Critical Curve

Taking into account equation (14) a motion equations are found by using Hamilton's principle from variation of energy integral

$$S = \int_{\mu_0}^{\mu_1} L d\mu = -\int_{\mu_0}^{\mu_1} \rho d\mu, \tag{23}$$

where μ_0, μ_1 are values of the affine parameter in points, which are linked by found extremal curve. Energy ρ is non-zero, its variations leave interval to be light-like, and application of standard variational procedure yields Euler-Lagrange equations

$$\frac{d}{d\mu}\frac{\partial\rho}{\partial u^\lambda} - \frac{\partial\rho}{\partial x^\lambda} = 0. \tag{24}$$

Equations of isotropic critical curve can be rewritten in form

$$\frac{dp_\lambda}{d\mu} - F_\lambda = 0. \tag{25}$$

Critical curve equations are obtained by substitution of partial derivatives (12) and (13) in these equations. For derivative of the first component of four-velocity vector we have

$$\frac{du^1}{d\mu} + \frac{u^1}{2u_1}\frac{\partial g_{ij}}{\partial x^1}u^i u^j = 0. \tag{26}$$

In the general form, the equations (25) will be

$$\begin{gathered}(g_{1k}v_\lambda - g_{k\lambda}u_1)\frac{du^k}{d\mu} + \\ \left[\left(\frac{\partial g_{1j}}{\partial x^i} - \frac{u_1}{2u^1}\frac{du^1}{d\mu}\right)u_\lambda - \left(\frac{\partial g_{\lambda i}}{\partial x^j} - \frac{1}{2}\frac{\partial g_{ij}}{\partial x^\lambda}\right)u_1\right]u^i u^j = 0.\end{gathered} \tag{27}$$

Replacement of derivative $du^1/d\mu$ here on its expression obtained from (26) gives

$$\begin{gathered}(g_{1k}u_\lambda - g_{k\lambda}u_1)\frac{du^k}{d\mu} + \\ \left[\left(\frac{\partial g_{1j}}{\partial x^i} - \frac{1}{2}\frac{\partial g_{ij}}{\partial x^1}\right)u_\lambda - \left(\frac{\partial g_{\lambda i}}{\partial x^j} - \frac{1}{2}\frac{\partial g_{ij}}{\partial x^\lambda}\right)u_1\right]u^i u^j = 0.\end{gathered} \tag{28}$$

The components of these equations can be expressed in terms of the components of the geodetic equations

$$\frac{d^2x^l}{d\mu^2}+\Gamma^l_{ij}\frac{dx^i}{d\mu}\frac{dx^j}{d\mu}=0, \tag{29}$$

where Γ^l_{ij} are Christoffel symbols:

$$\Gamma^l_{ij}=\frac{1}{2}g^{lm}\left(\frac{\partial g_{jm}}{\partial x^i}+\frac{\partial g_{im}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^m}\right). \tag{30}$$

After multiplication by g_{kl} and summations over repeated indexes l the geodesics equations (29) will be following:

$$g_{kj}\frac{d^2x^j}{d\mu^2}+\frac{1}{2}\left(\frac{\partial g_{ki}}{\partial x^j}+\frac{\partial g_{kj}}{\partial x^i}-\frac{\partial g_{ji}}{\partial x^k}\right)\frac{dx^j}{d\mu}\frac{dx^i}{d\mu}=0. \tag{31}$$

We denote the left-hand side of this equation by $\overset{\frown}{D}_k$. Then the equations (28) are written in the form

$$u_\lambda\overset{\frown}{D}_1-u_1\overset{\frown}{D}_\lambda=0. \tag{32}$$

Coupled with equation (26), they describe the motion of a light-like particle in accordance with the principle of an extremal energy integral of like-light particle.

5 Energy and Momentum of Particle Transferred to Gravity Field

Euler-Lagrange equations can be rewritten in form

$$\frac{dp_\lambda}{d\mu}-F_\lambda=0\quad . \tag{33}$$

Passing in these equations to the associated canonical momenta and forces, we obtain

$$F^k=\frac{dp^k}{d\mu}+g^{k\lambda}\frac{dg_{\lambda i}}{d\mu}p^i. \tag{34}$$

In accordance with conservation laws, the vector of energy and momentum of a system that includes a particle and the gravitational field generated by it, denoted by $\bar{p}^k$, can be written as the sum of the momentum and energy of the particle itself $p^k$and transmitted it to the gravitational field $\overset{\leftrightarrow}{p}^k$. The vector $\bar{p}^k$ changes under the influence of the force from the source of gravity:

$$\frac{d\bar{p}^k}{d\mu}=\frac{dp^k}{d\mu}+\frac{d\overset{\leftrightarrow}{p}^k}{d\mu}=F^k. \tag{35}$$

Comparing two expressions for F^k and passing in (34) to the partial derivatives of metrical coefficients we find the rate of exchange of energy and momentum between particle and gravitational field

$$\frac{d\overset{\leftrightarrow}{p}^k}{d\mu}=g^{k\lambda}\frac{\partial g_{\lambda i}}{\partial x^j}u^jp^i. \tag{36}$$

From the conservation laws it follows that the force acting on the system including the particle and the gravitational field generated by it is equal in magnitude and opposite in sign to the force acting on the system of the source of gravitation from the side of the particle system. This is equivalent to fulfilling Newton's third law. Its adherence to the Newtonian limit of gravity means the equality of the passive and active gravitational masses.

6 Identity of Geodesics and Extreme Energy Integral Curves in Static Space-Time

The principle of the stationary energy integral of a light-like particle looks for an extremal on a set of isotropic paths. A null geodesic, obtained by a null path variation, is an extremal on a set of isotropic and non-isotropic paths whose value approaches zero [4, 5, 6].

For the isotropic paths a transformation [11] to metric $\overline{g}_{ij} = g_{ij}/g_{00}$ is equivalent to replacement of parameter μ on $d\overline{\mu} = d\mu/\sqrt{g_{00}}$, to which the four-velocities $\overline{u}^i = dx^i/d\overline{\mu}$ correspond. The curve of motion of lightlike particle in four-dimensional space-time and value of energy ρ are invariant under this reparametrization. For the static spacetime the first equation of motion with appropriate parameter $\overline{\mu}$ gives $\overline{u}^0 = 1$. Canonical momentum and forces take form

$$\overline{p}_\lambda = \overline{u}_\lambda; \qquad \overline{F}_\lambda = \frac{1}{2}\frac{\partial \overline{g}_{ij}}{\partial x^\lambda}\overline{u}^i\overline{u}^j. \tag{37}$$

Substitution of them in Euler-Lagrange equations gives

$$\frac{d}{d\mu}\left(\overline{g}_{\lambda k}\overline{u}^k\right) = \frac{1}{2}\frac{\partial \overline{g}_{ij}}{\partial x^\lambda}\overline{u}^i\overline{u}^j. \tag{38}$$

After performing the differentiation on the left-hand side of equations and multiplying them by $\overline{g}^{l\lambda}$ this expression, the summation over the repeated index λ yields null geodesic equations (29). So in case of the static spacetime the geodesic principle and the energy variational method as well as Fermat's principle give the same solution for the light propagation. So in case of the static spacetime the geodesic principle and the energy variational method as well as Fermat's principle give the same solution for the light propagation.

7 Comparison of Null Geodesics, Energy Integral Variation and Fermat principles

Let us clear whether proposed variational method conforms to Fermat's principle for stationary gravity field [2], which is formulated as follows

$$\delta \int \frac{1}{g_{11}}\left(dl + g_{1k}dx^k\right) = 0, \tag{39}$$

where dl is element of spatial distance along the ray

$$dl^2 = \left(\frac{g_{1p}g_{1q}}{g_{11}} - g_{pq}\right)dx^p dx^q. \tag{40}$$

Denoting

$$df = \frac{1}{g_{11}}\left(dl + g_{1k}dx^k\right), \tag{41}$$

and comparing this expression with (10) we write

$$\frac{df}{d\mu} = \rho u^1. \tag{42}$$

Therefore, variation (39) is equivalent to variation of integral

$$S_1 = \int_{\mu_0}^{\mu_1} \rho u_f^1 d\mu. \tag{43}$$

For stationary gravity field the metrical coefficients doesn't depend on time so equation of motion (26) gives constant velocity u^1 and partial derivative (11) will be $\partial\rho/\partial x^1 = 0$. The Euler-Lagrange equations of Fermat's principle given by action (43) and corresponded to the time coordinate are

$$\frac{d\rho}{d\mu} - \frac{\partial \rho}{\partial x^1}u_f^1 = 0. \tag{44}$$

Thus, differential of energy ρ is zero which is consistent with the initial condition of its equality to 1 along critical curve and this expression is idential equation. For the space coordinates, the equations, given by Fermat's principle, are follows:

$$\frac{d}{d\mu}\left(\frac{\partial\rho}{\partial u_f^k}\right)u_f^1+\frac{\partial\rho}{\partial u_f^k}\frac{du_f^1}{d\mu}-\frac{\partial\rho}{\partial x^k}u_f^1=0. \quad (45)$$

Together with condition (9) we have four equations for five unknown variables x^i,μ. This allows us to select μ so that the velocity u_f^1 is constant and the second term in the left part of equations for the space coordinates will be vanishing and they shall be identical to (25). Hence, energy integral variation and Fermat principles give identical curves in stationary space-times.

In [12] the generalized Fermat's principle is proposed and it is shown that obtained curves are null geodesics. The time is used as a functional and together as a variable. It is applied Pontryagin's minimum principle of the optimal control theory and obtained an effective Hamiltonian for the light-like particle motion in a curved spacetime. The dynamical equations for this Hamiltonian are

$$Q=u^1 \quad (46)$$

and

$$dd\mu(\partial Q\partial\dot{x}^q)-\partial Q\partial x^q-\partial Q\partial x^1\partial Q\partial\dot{x}^q=0\ . \quad (47)$$

Function Q coincides with $-df/d\mu$, under condition that the metric coefficients in (40), (41) also depend on time. It is proposed that obtained dynamical equations correspond to the null geodesic. Following from (42) expression for energy $\rho=Q/u^1$ substituted in Euler-Lagrange equations (24) yields

$$\frac{1}{u^1}\frac{d}{d\mu}\left(\frac{\partial Q}{\partial u^q}\right)-\frac{1}{(u^1)^2}\frac{\partial Q}{\partial u^q}\frac{du^1}{d\mu}-\frac{1}{u^1}\frac{\partial Q}{\partial x^q}=0. \quad (48)$$

Comparing equations (26) and (11) we write

$$\frac{du^1}{d\mu}=(u^1)^2\frac{\partial(Q/u^1)}{\partial x^1}=u^1\frac{\partial Q}{\partial x^1}. \quad (49)$$

Substituting this expression in (48) and multiplying it by u^1 gives equations (47), which confirms the identity of principle of an extremal energy integral of light-like particle and generalized Fermat's principle.

8 Photon's Dynamics in Schwarzschild Space-Time

8.1 Spherical Coordinates

A centrally symmetric gravity field in the free space is described by the Schwarzschild metric. At spherical coordinates $x^i=(\tau,r,\theta,\varphi)$ with $\tau=ct$ its line element is

$$ds^2=\left(1-\frac{\alpha}{r}\right)d\tau^2-\left(1-\frac{\alpha}{r}\right)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta d\phi^2), \quad (50)$$

where α is constant. To find the photon motion, we solve the Euler-Lagrange equations, which for static metrics give for Lagrangian (14) a solution that is identical with the geodesics. In plane $\theta=\pi/2$ equations (24) with canonical momenta (16) and forces (17) yields [4, 5, 6]:

$$\frac{d\tau}{d\mu}=1, \quad (51)$$

$$\frac{d\varphi}{d\mu}=\frac{B}{r^2}\left(1-\frac{\alpha}{r}\right), \quad (52)$$

where B is constant. Substituting these velocities in equation $ds=0$ we find

$$\frac{dr}{d\mu}=\pm\left[\left(1-\frac{\alpha}{r}\right)^2-\left(\frac{B}{r}\right)^2\left(1-\frac{\alpha}{r}\right)^3\right]^{1/2}. \quad (53)$$

The value of the coordinate velocity in the remote frame is

$$v = \sqrt{\left(r\frac{d\varphi}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2} = c\left(1 - \frac{\alpha}{r}\right). \tag{54}$$

In the framework of geometrical optics using analogy its analogy with gravity [7, 8, 9, 13] the refraction index (6) is given by:

$$n = \left(1 - \frac{\alpha}{r}\right)^{-1}. \tag{55}$$

Turning to ESM we write four-momentum after rotation in the plane (TS) in space $G(1,4)$ (5):

$$\left(\frac{E}{c}, P, p_s\right) = \left(\frac{\hbar\omega}{c\left(1-\frac{\alpha}{r}\right)}, \frac{\hbar\omega}{c}, \frac{\hbar\omega}{c}\frac{[\alpha(2r-\alpha)]^{1/2}}{r-\alpha}\right). \tag{56}$$

In 4D space-time for the Schwarzschild field the canonical momenta are

$$p_1 = 1, \qquad 2 = \mp\frac{1}{\left(1-\frac{\alpha}{r}\right)}\sqrt{1 - \frac{B^2}{r^2}\left(1 - \frac{\alpha}{r}\right)}, \tag{57}$$

$$p_3 = 0, \qquad p_4 = -B. \tag{58}$$

Nonzero components of the contravariant vector of momenta are given by

$$p^1 = \left(1 - \frac{\alpha}{r}\right)^{-1}, \tag{59}$$

$$p^2 = \pm\sqrt{1 - \frac{B^2}{r^2}\left(1 - \frac{\alpha}{r}\right)}, \tag{60}$$

$$p^4 = \frac{B}{r^2}. \tag{61}$$

The physical energy and momentum are matched exactly with the contravariant vector, since in the limit of the Minkovsky space it has momentum components with a sign coinciding with the direction of motion.

8.2 Rectangular Coordinates

To determine the magnitude of the photon momentum we use the Schwarzschild metric in rectangular coordinates [5, 6]. To the isotropic form of metric one can go from its spherical form (??) with the help of the transformation

$$r = \left(1 + \frac{\alpha}{4\bar{r}}\right)^2 \bar{r}, \tag{62}$$

and it is written as

$$ds^2 = c^2\left(\frac{1-\frac{\alpha}{4\bar{r}}}{1+\frac{\alpha}{4\bar{r}}}\right)^2 dt^2 - \left(1 + \frac{\alpha}{4\bar{r}}\right)^4 (dx^2 + dy^2 + \quad dz^2), \tag{63}$$

where (t, x, y, z) is rectangular frame and $\bar{r} = \sqrt{x^2 + y^2 + z^2}$.

We will consider the motion in the plane $z = 0$ and seek the force acting on the particle at a point $(t, x, 0,0)$ that corresponds to the value of the angular coordinate $\varphi = 0$ in the spherical frame. Coordinate transformations in the plane are

$$x = \bar{r}\cos\varphi, \qquad y = \bar{r}\sin\varphi. \tag{64}$$

The nonzero spatial components of the 4-velocity are

$$\bar{u}^2 = \frac{dx}{d\mu} = \frac{d\bar{r}}{d\mu}, \qquad \bar{u}^3 = \frac{dy}{d\mu} = \frac{d\varphi}{d\mu}\bar{r}. \tag{65}$$

The transformation (62) implies the relation

$$dr = \left(1 - \frac{\alpha^2}{16\bar{r}^2}\right) d\bar{r}. \tag{66}$$

Equations (51)-(53) yield

$$\bar{u}^1 = 1, \tag{67}$$

$$\bar{u}_1 = \left(\frac{1-\frac{\alpha}{4\bar{r}}}{1+\frac{\alpha}{4\bar{r}}}\right)^2, \tag{68}$$

$$\bar{u}^2 = \pm\frac{\left(1-\frac{\alpha}{4\bar{r}}\right)}{\left(1+\frac{\alpha}{4\bar{r}}\right)^3}\left[1 \quad -\frac{B^2\left(1-\frac{\alpha}{4\bar{r}}\right)^2}{\bar{r}^2\left(1+\frac{\alpha}{4\bar{r}}\right)^6}\right]^{1/2}, \tag{69}$$

$$\bar{u}^3 = \frac{B\left(1-\frac{\alpha}{4\bar{r}}\right)^2}{\bar{r}\left(1+\frac{\alpha}{4\bar{r}}\right)^6}. \tag{70}$$

Substitution of these velocities in (18) gives components of associated vector of the canonical momenta

$$\bar{p}^1 = \left(\frac{1+\frac{\alpha}{4\bar{r}}}{1-\frac{\alpha}{4\bar{r}}}\right)^2, \tag{71}$$

$$\bar{p}^2 = \pm\frac{1}{\left(1-\frac{\alpha}{16\bar{r}^2}\right)}\left[1 \quad -\frac{B^2\left(1-\frac{\alpha}{4\bar{r}}\right)^2}{\bar{r}^2\left(1+\frac{\alpha}{4\bar{r}}\right)^6}\right]^{1/2}, \tag{72}$$

$$\bar{p}^3 = \frac{B}{\bar{r}\left(1+\frac{\alpha}{4\bar{r}}\right)^4}. \tag{73}$$

Passing back from the variable $\bar{r}$ to r, we write, in accordance with equation (20), the value of the photon energy and momentum in a remote coordinate frame

$$E = \hbar\omega_0\left(1-\frac{\alpha}{r}\right)^{-1}, \tag{74}$$

$$\bar{P} = [(\bar{p}^2) + (\bar{p}^3)]^{1/2} = \frac{1}{\left(1-\frac{\alpha}{16\bar{r}^2}\right)}\frac{\hbar\omega_0}{c}, \tag{75}$$

where ω_0 is the photon frequency at infinity at the world line with unlimited r. Moving to the scale of the length of spherical frame in view of Eq. (66) we obtain $P = \hbar\omega_0/c$. By defining ω in the same way we obtain the coincidence of the energy and momentum (56) in the embedded four-dimensional space-time in ESM with the result given by variational principle of the stationary energy integral of photon.

8.3 Forces and Gravity Mass of Photon

At spherical coordinates the canonical forces (17) are

$$F_1 = F_3 = F_4 = 0, \tag{76}$$

$$F_2 = \frac{\alpha}{r^2\left(1-\frac{\alpha}{r}\right)} - \frac{B^2}{r^3} + \frac{\alpha B^2}{2r^4}. \tag{77}$$

A nonzero component of vector associated with the canonical forces is

$$F^2 = -\frac{\alpha}{r^2} + \frac{B^2}{r^3}\left(1-\frac{\alpha}{r}\right)\left(1-\frac{\alpha}{2r}\right). \tag{78}$$

In so far as with gravitational constant G and active gravitational mass M the Newtonian limit of gravity theory requires $\alpha = 2GM$, for the radial motion $(B = 0)$ the first term of F^2 yields (22) twice Newton gravity force acting on a photon

$$Q^2 = -\hbar\omega_0\frac{\alpha}{r^2}. \tag{79}$$

It corresponds to the passive gravitatinal mass of the photon

$$m_{gp} = 2\hbar\omega_0. \quad (80)$$

Considering the non-radial motion in order to avoid the appearance of a fictitious component of the force due to the sphericity of the coordinate system, we use the Schwarzschild metric in rectangular coordinates (63). The components of the canonical forces vector F^k (21) are put in correspondence with the gravitational forces acting on the particle. Substituting nonzero 4-velocity components (67)-(70) in (17), we find the unique nonzero component of the force vector (22) acting on the photon:

$$\overline{Q}^2 = - - \hbar\omega_0 \frac{\alpha\left(1-\frac{\alpha}{8\overline{r}}\right)}{\overline{r}^2\left(1+\frac{\alpha}{4\overline{r}}\right)^5\left(1-\frac{\alpha}{4\overline{r}}\right)}. \quad (81)$$

Taking into account transformation (62) it is rewritten as

$$\overline{Q}^2 = - - \hbar\omega_0 \frac{\alpha\left(1-\frac{\alpha}{8\overline{r}}\right)}{r^2\left(1-\frac{\alpha^2}{16\overline{r}^2}\right)}. \quad (82)$$

Its magnitude does not depend on the direction of motion of the photon. This formula differs from force in spherical coordinates (78) because the expression for the canonical force (17) is non-covariant, that is, with this approach gravity force acting on the photon depends on the choice of the coordinate system. However, in the limit of weak gravity these expressions asymptotically converge and give Newton's law of gravitation with passive gravitational mass of the photon $2\hbar\omega_0$ (80). One conforms to the light deflection in central gravity field, which is twice value being given by the Newton gravity theory.

Obtained gravitational mass of the light-like particle is independent on the direction of its motion. The gravitational mass of a photon for low gravity is equal to doubled mass of a material particle, equivalent to its energy. This corresponds to the result of Tolman [14] for active gravity mass of photon. He solved Einstein's equations for electromagnetic field in case of the weak gravitation and obtained it for the interaction between a light package or beam and a material particle.

This result can have the following application. At annihilation of an electron and positron the energy determined from non-gravitational interactions and the momentum are preserved. We will consider how the gravitational mass of system changes. Although it is not known exactly whether the gravitational mass of the positron is positive or negative, some estimates give its positive value [15]. Proceeding from this assumption the total gravitational mass of an electron and positron $2m_e$ is twice less than the gravitational mass of the formed gamma quanta $4m_e$. This raises the question of mass conservation [16]. If to consider energy as a gravitation source, it means that on condition of its preservation at annihilation besides gamma quanta this process has to be allocated the particles g^- which are carrying away negative energy as a source of gravitational field, that is, having negative gravitational mass. Process of annihilation will look as follows

$$e^+ + e^- \to 2\gamma + 2g^-. \quad (83)$$

The particles g^- with gravitational mass

$$m_{g\_} = -m_e \quad (84)$$

do not have a kinetic momentum and therefore their detection by standard means of particle registration, for example, a bubble chamber, is not possible. However, if there is opposite to them in "a gravitational charge" particle g^+, it can apply for a share in dark energy and matter, as well as g^-.

9 Extremal Isotropic Curves in Goedel Space-Time

The stationary solution of the Einstein's field equation with cosmological constant found by Goedel describes gravity field of the rotating uniform dust matter. With coordinates $x^i = (t, r, y, z)$ the line element is written in form

$$ds^2 = dt^2 - dr^2 - dz^2 + 2\exp(\sqrt{2}\omega r)dtdy + \frac{1}{2}\exp(2\sqrt{2}\omega r)dy^2, \tag{85}$$

where ω is constant.

9.1 Solution by Use of Principle of Stationary Integral of Energy

The canonical momenta (16) for cyclic coordinates t, y, z are the constants of motion [5, 6]. They are written in form

$$p_1 = \frac{1}{u^1}, \tag{86}$$

$$p_3 = \frac{\exp(\sqrt{2}\omega r)u^1 + \frac{1}{2}\exp(2\sqrt{2}\omega r)u^3}{u^1(u^1 + \exp(\sqrt{2}\omega r)u^3)}, \tag{87}$$

$$p_4 = -\frac{u^4}{u^1(u^1 + \exp(\sqrt{2}\omega r)u^3)}. \tag{88}$$

These equations with following from Eq. (85) condition

$$0 = (u^1)^2 - (u^2)^2 - (u^4)^2 + 2\exp(\sqrt{2}\omega r)u^1u^3 + \frac{1}{2}\exp(2\sqrt{2}\omega r)(u^3)^2 \tag{89}$$

yield components of the four-velocity vector:

$$\frac{dt}{d\mu} = \frac{1}{p_1}, \tag{90}$$

$$\frac{dr}{d\mu} == \pm\frac{\left[4p_1p_3\exp(\sqrt{2}\omega r) - (p_1^2 + p_4^2)\exp(2\sqrt{2}\omega r) - 2p_3^2\right]^{1/2}}{p_1(p_1\exp(\sqrt{2}\omega r) - 2p_3)}, \tag{91}$$

$$\frac{dy}{d\mu} = 2\frac{p_3 - p_1\exp(\sqrt{2}\omega r)}{p_1\exp(\sqrt{2}\omega r)(p_1\exp(\sqrt{2}\omega r) - 2p_3)}, \tag{92}$$

$$\frac{dz}{d\mu} = \frac{p_4\exp(\sqrt{2}\omega r}{p_1(p_1\exp\sqrt{2}\omega r - 2p_3)}. \tag{93}$$

With $p_1\exp(\sqrt{2}\omega r) = 2p_3$ the singularity takes place.

The canonical momentum corresponding to coordinate r is

$$p_2 = \pm\left[4p_1p_3\exp(-\sqrt{2}\omega r) - (p_1^2 + p_4^2) - 2p_3^2\exp(-2\sqrt{2}\omega r)\right]^{1/2}. \tag{94}$$

Canonical forces have values

$$F_1 = F_3 = F_4 = 0, \tag{95}$$

$$F_2 = 2\sqrt{2}\omega\frac{p_3(p_3 - p_1\exp(\sqrt{2}\omega r))}{(p_1\exp(\sqrt{2}\omega r) - 2p_3)^2}. \tag{96}$$

Associated canonical momentum and forces are

$$p^1 = -p_1 + 2p_3\exp(-\sqrt{2}\omega r), \tag{97}$$

$$p^2 = \mp\left[4p_1p_3\exp(-\sqrt{2}\omega r) - (p_1^2 + p_4^2) - 2p_3^2\exp(-2\sqrt{2}\omega r)\right]^{1/2}, \tag{98}$$

$$p^3 = 2p_1\exp(-\sqrt{2}\omega r) - 2p_3\exp(-2\sqrt{2}\omega r), \tag{99}$$

$$p^4 = -p_4; \tag{100}$$

$$F^1 = F^3 = F^4 = 0, \tag{101}$$

$$F^2 = -2\sqrt{2}\omega \frac{p_3\left(p_3 - p_1 \exp(\sqrt{2}\omega r)\right)}{\left(p_1 \exp(\sqrt{2}\omega r) - 2p_3\right)^2}. \tag{102}$$

9.2 Comparison of Extreme Integral of Energy Curves and Geodesics

The procedure for obtaining the geodesic equations [17] by the variation of the integral of expression

$$\eta = g_{ij} \frac{dx^i}{d\mu} \frac{dx^j}{d\mu}, \tag{103}$$

is identical to finding the Euler-Lagrange equations [10] for the Lagrangian

$$L_g = \frac{1}{2}\eta, \tag{104}$$

that is, these equations are identical. For metric (85) we have

$$L_g = \frac{1}{2}\left[(\tilde{u}^1)^2 - (\tilde{u}^2)^2 - (\tilde{u}^4)^2 + 2\exp(\sqrt{2}\omega r)\tilde{u}^1\tilde{u}^3 + \frac{1}{2}\exp(2\sqrt{2}\omega r)(\tilde{u}^3)^2\right], \tag{105}$$

where $\tilde{u}^i$ are 4-velocities of geodesics. The constants of motion are the canonical momenta

$$\tilde{p}_\lambda = \frac{\partial L_g}{\partial u^\lambda} \tag{106}$$

corresponding to cyclic coordinates

$$\tilde{p}_1 = \tilde{u}^1 + \exp(\sqrt{2}\omega r)\tilde{u}^3,$$

$$\tilde{p}_3 = \exp(\sqrt{2}\omega r)\tilde{u}^1 + \frac{1}{2}\exp(2\sqrt{2}\omega r)\tilde{u}^3,$$

$$\tilde{p}_4 = -\tilde{u}^4. \tag{107}$$

These equations, together with condition (89) for 4-velocities $\tilde{u}^i = \partial x^i / \partial \tilde{\mu}$ with affine parameter $\tilde{\mu}$ yield

$$\frac{cdt}{d\tilde{\mu}} = -\tilde{p}_1 + 2\tilde{p}_3 \exp(-\sqrt{2}\omega r), \tag{108}$$

$$\frac{dr}{d\tilde{\mu}} = \pm\left[-\tilde{p}_1^2 - \tilde{p}_4^2 + 4\tilde{p}_1\tilde{p}_3\exp(-\sqrt{2}\omega r) - 2\tilde{p}_3^2\exp(-2\sqrt{2}\omega r)\right]^{1/2}, \tag{109}$$

$$\frac{dy}{d\tilde{\mu}} = 2\left[\tilde{p}_1\exp(-\sqrt{2}\omega r) - \tilde{p}_3\exp(-2\sqrt{2}\omega r)\right], \tag{110}$$

$$\frac{dz}{d\tilde{\mu}} = -\tilde{p}_4. \tag{111}$$

As a result, after substitution $\tilde{p}_i^* = \tilde{p}_i / \tilde{p}_1$ in case $\tilde{p}_1 \neq 0$ we obtain velocities as the derivatives of spatial coordinates with respect to time

$$\dot{r}_g = \pm \frac{\left[-(1+\tilde{p}_4^{*})^2)\exp(2\sqrt{2}\omega r) + 4\tilde{p}_3^*\exp(\sqrt{2}\omega r) - 2(\tilde{p}_3^*)^2\right]^{1/2}}{\exp(\sqrt{2}\omega r) - 2\tilde{p}_3^*}, \tag{112}$$

$$\dot{y}_g = 2\frac{\tilde{p}_3^* - \exp(\sqrt{2}\omega r)}{\exp(\sqrt{2}\omega r)\left[\exp(\sqrt{2}\omega r) - 2\tilde{p}_3^*\right]}, \tag{113}$$

$$\dot{z}_g = \frac{\tilde{p}_4^* \exp(\sqrt{2}\omega r)}{\exp(\sqrt{2}\omega r) - 2\tilde{p}_3^*}. \tag{114}$$

Solution by use of principle of stationary integral of energy (90)-(93) after substitution $p_i^* = p_i/p_1$ in case $p_1 \neq 0$ gives velocities

$$\dot{r} = \pm \frac{\left[(-1+(p_4^*)^2)\exp(2\sqrt{2}\omega r) + 4p_3^* \exp(\sqrt{2}\omega r) - 2(p_3^*)^2\right]^{1/2}}{\exp(\sqrt{2}\omega r) - 2p_3^*}, \tag{115}$$

$$\dot{y} = 2\frac{p_3^* - \exp(\sqrt{2}\omega r)}{\exp(\sqrt{2}\omega r)\left[\exp(\sqrt{2}\omega r) - 2p_3^*\right]}, \tag{116}$$

$$\dot{z} = \frac{p_4^* \exp(\sqrt{2}\omega r)}{\exp(\sqrt{2}\omega r) - 2p_3^*}. \tag{117}$$

Expressions (112)-(114) and (115)-(117) are identical. Thus, curves of extreme integral of a light-like particle energy coincide with isotropic geodesics in Goedel space-time. Since it was shown in §5 that the first of these methods yields solutions consistent with the Fermat principle, we can conclude that for this stationary but not static space the solution of geometric optics coincide with null geodesics.

Although expressions for generalized momenta (16) are not themselves covariant, they give values (100), which coincide with covariant momenta

$$p_g^j = g^{ij}\frac{\partial L_g}{\partial \tilde{u}^i} = \tilde{u}^j. \tag{118}$$

These momenta are expressed by equations (108)-(111).

10 Introduction of the photon 4-momentum via rotations in ESM generally

In addition to the (TS)-rotation (5) of 5-momenta [9]:

$$\frac{E'}{c} = \frac{E}{c}\cosh\varphi_{TS} + p_s \sinh\varphi_{TS}, \tag{119}$$

$$P' = P, \tag{120}$$

$$p'_s = p'\cosh\varphi_{TS} + \frac{E}{c}\sinh\varphi_{TS} \tag{121}$$

in ESM there is (XS)-rotation

$$\frac{E'}{c} = \frac{E}{c}, \tag{122}$$

$$P' = P\cosh\varphi_{XS} + p_s \sinh\varphi_{XS}, \tag{123}$$

$$p'_s = p'\cosh\varphi_{XS} + P\sinh\varphi_{XS}. \tag{124}$$

With the help of these transformations from the components of photon 5-momentum (5) in a flat extended space, one can pass to the components of its 4-momentum in an arbitrary 4-dimensional space (7):

$$\left(\frac{\hbar\omega}{c}, \frac{\hbar\omega}{c}, 0\right) \quad \rightarrow \quad \left(\frac{\hbar\omega}{c}\mathrm{F}^T(\mathrm{x}_i), \frac{\hbar\omega}{c}\mathrm{F}^P(\mathrm{x}_i), \frac{\hbar\omega}{c}\mathrm{F}^S(\mathrm{x}_i)\right), \tag{125}$$

where $\mathrm{F}^T(\mathrm{x}_i), \mathrm{F}^P(\mathrm{x}_i), \mathrm{F}^S(\mathrm{x}_i)$ are functions of coordinates. Transformations are not communicative at specified angles of rotation φ_{TS} and φ_{XS}:

$$(TS) - (XS) \neq (XS) - (TS). \quad (126)$$

In the case of a material particle, a transformation (TX) is added to them.

11 Conclusion

Canonical 4-momentum is the result given by variational principle of the stationary energy integral of photon. The physical energy and momentum of photon are matched exactly with the contravariant 4-momentum, since in the limit of the Minkovsky space it has momentum components with a sign coinciding with the direction of motion. This approach applied to the Schwarzschild space-time and Extended Space Model are compared. In ESM these energy and momentum in the embedded four-dimensional space-time are obtained by (TS)-turn that is corresponded to the photon's motion in space with refraction index $n > 1$. With the help of (TS) and (XS) transformations from the components of photon 5-momentum in a flat extended space, one can pass to the components of its 4-momentum in an arbitrary 4-dimensional space. A transformation (TX) is added to them In the case of a material particle.

The identity of the generalized Fermat principles and the stationary energy integral of a light-like particle is proved. The virtual displacements of coordinates retain path of the light-like particle to be null in the pseudo-Riemann space-time, i.e. not lead to the Lorentz-invariance violation in locality and correspond to the variational principles of mechanics. The equivalence of the solutions given by the first principle, to the geodesics, means that the use of the second also turns out geodesics. The stationary energy integral principle gives a system of equations that has one equation more. This makes it possible to uniquely determine the affine parameter and energy-momentum vector of the particle.

A definite Lagrangian produces particle canonical momenta and forces acting on it in the coordinate frame. Contravariant forces are mapped to the components of the vector of the gravitational force. The four-force vector is not covariant. The value of the force acting on a particle depends on the choice of the coordinate frame, and therefore the quantities determined through them are meaningful only for weak gravity, for which its values asymptotically converge in the different coordinate frames. The analogy between the mechanics of particle motion in the Schwarzschild space and Newton's gravity theory allows to determine passive gravitational mass of the photon, which is equal to twice the mass of a material particles of the same energy determined from non-gravitational interactions. This corresponds to the result of Tolman for active gravity mass of photon. This discrepancy suggests that at annihilation of an electron and positron in addition to gamma quanta particles are released that have zero kinetic energy and momentum and carrying away negative energy as a source of gravitational field, that is, they have negative gravitational mass.

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