MODELING THE UNFAIR TOSS OF AN UNBIASED COIN

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The torque-free motion of a coin in gravitational field is analyzed for general initial conditions. After launch at the end of motion the coin lands on a soft surface that allows no bouncing. The model assumes the coin is a thin cylindrical rigid body and the air resistance is absent. The motion is described by numerical integration of the Euler equation of motion. Biased probabilities of heads and tails are obtained for various initial conditions accessible to a skilled person.

Key words: coin toss, probability of heads, rigid body, dynamics equations .

1. INTRODUCTION

Coin tossing is the go-to random mechanism when choosing between two equally probable outcomes. However, coin tossing is a phenomenon that obeys the laws of mechanics and thus a deterministic process which can be mathematically modeled [2, 6–10]. The position of the coin at a moment is determined by the launch conditions, that is by the initial conditions. Though predictable by definition, the classical mechanics models may still reveal interesting physics. From the large category of examples, some involve conservation or balance laws of the angular momentum, linear momentum or energy. Thus a certain rule of energy dissipation in the oscillatory motion of a gravitational pendulum [4] generates a band-like structure of the number of collisions as function of the initial amplitude and damping, and establishes an unexpected connection with the solid state physics quantum models of the materials. Unforeseen constant of motion found by applying the momentum conservation law to the idealized motion of jetting paralarvae moving in sea water in vertical direction [5] is another example from a large list. In this didactic paper, we analyze the flight of a coin using such a mathematical model. We start by solving the deterministic motion of a coin assimilated with a uniform cylindrical body. Differently from the treatment in Ref. [1] we will consider that the coin has a more complex motion with, in addition to the rotation about an axis situated in the horizontal plane, an initial angular velocity along the vertical axis. Such a coin toss, where there are non-zero initial angular velocities in all three spatial directions may be obtained by a skilled person. We determine the orientation of the coin in flight by computing the dihedral angle between the initial and at a moment plane of the coin. Regarding the simulation of the random process, we consider the range of initial conditions a source of uncertainty.

The paper is structured as follows. In section 2, the theory of the deterministic torque-free coin motion in gravitational field is briefly presented. In section 3, we introduce the models for computing the random toss and discuss the simulation results for some natural initial conditions. In section 4, we present conclusions.

2. THE PHYSICAL SETUP

We begin by stating our main assumptions. Firstly, we assume that the equations of motion are the Newton equations, with no external source of influence, such as fluctuations of air, thermodynamic or quantum fluctuations of the coin, etc. We also neglect the air drag. We consider that the coin lands on a soft surface that allows no bouncing, such as the palm of the hand.

To derive the equations of motion it is convenient to use three reference frames: the laboratory frame (LF), XYZ, the body frame (BF), xyz, and a transport frame (TF), $x_1y_1z_1$ (see figure 1) [1]. The moment of inertia tensor has the diagonal form in the BF:

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}.$$

The components of the moment of inertia tensor with respect to the principal axes are as follows:

$$I_1 = I_2 = I_{xx} = I_{yy} = \frac{m}{4}(r^2 + \frac{h^2}{3}), I_3 = I_{zz} = \frac{mr^2}{2},$$

where r is the radius, h is the thickness, and m is the mass of the cylinder.
We consider that the coin is vertically launched and consequently the center of mass (CM) of the body has a vertical motion in gravitational field with velocity:

$$v_z = v_0 - gt,$$

where g is the gravitational acceleration and t is time. Following Ref. [1], we can write Euler's equations in the BF as follows:

$$\begin{aligned} I_1\dot{\omega}_x + (I_3 - I_1)\omega_y\omega_z &= 0, \\ I_1\dot{\omega}_y + (I_1 - I_3)\omega_x\omega_z &= 0, \\ I_3\dot{\omega_z} &= 0. \end{aligned} \tag{1}$$

The solutions of equations (1) are of the form:

$$\omega_x = \Omega\sin(qt+\delta), \omega_y = -\Omega\cos(qt+\delta). \tag{2}$$

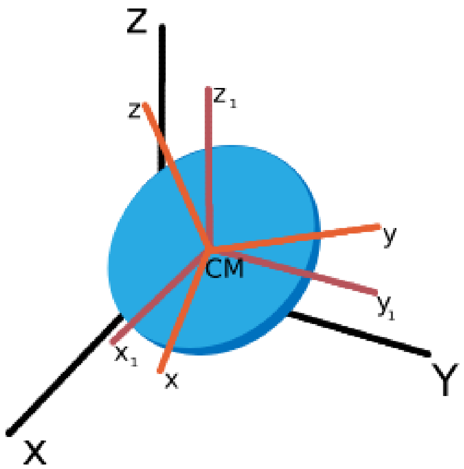


Fig. 1 – LF, TF, and BF are represented

where $q = (I_3 - I_1)\omega_{z0}/I_1$ and according to the initial conditions:

$$\Omega = \sqrt{\omega_{x0}^2 + \omega_{y0}^2}, \tan\delta = -\frac{\omega_{x0}}{\omega_{y0}}. \tag{3}$$

As ω_{z0} is assumed non-zero, one has $q \neq 0$, and from equations (2) and (3) one obtains:

$$\omega_x = \Omega\sin(qt+\delta), \omega_y = -\Omega\cos(qt+\delta), \tag{4}$$

with

$$\Omega = \sqrt{\omega_{x0}^2 + \omega_{y0}^2}, \tan\delta = -\frac{\omega_{x0}}{\omega_{y0}}, q = \frac{(I_3 - I_1)\omega_{z0}}{I_1}.$$

We introduce the Euler angles by definition [3] to describe the motion in the TF:

$$\begin{aligned} \omega_x &= \dot{\beta}\cos\gamma + \dot{\alpha}\sin\beta\sin\gamma, \\ \omega_y &= -\dot{\beta}\sin\gamma + \dot{\alpha}\sin\beta\cos\gamma, \\ \omega_{z0} &= \dot{\gamma} + \dot{\alpha}\cos\beta. \end{aligned} \tag{5}$$

To solve equations (5), we substitute $\dot{\alpha} = (\omega_{z0} - \dot{\gamma})/\cos\beta$ and obtain the following set of differential equations:

$$\omega_x = \dot{\beta}\cos\gamma + (\omega_{z0} - \dot{\gamma})\tan\beta\sin\gamma, \tag{6}$$

$$\omega_y = -\dot{\beta}\sin\gamma + (\omega_{z0} - \dot{\gamma})\tan\beta\cos\gamma. \tag{7}$$

Next, we multiply equation (6) by $\cos\gamma$ and subtract equation (7) multiplied by $\sin\gamma$ to obtain:

$$\dot{\beta} = \omega_x\cos\gamma - \omega_y\sin\gamma. \tag{8}$$

Multiplying equation (6) by $\sin\gamma$ and adding the product of equation (7) by $\cos\gamma$ we obtain:

$$\tan\beta = \frac{\omega_x \sin\gamma + \omega_y \cos\gamma}{\omega_{z0} - \dot{\gamma}}. \tag{9}$$

Manipulating equations (8) and (9) we obtain the differential equation for γ:

$$\ddot{\gamma} + A(\gamma,t)\dot{\gamma} + B(\gamma,t)\dot{\gamma}^2 + C(\gamma,t) = 0. \tag{10}$$

Where A,B, and C are functions of γ and t:

$$\begin{aligned}
A(\gamma,t) &= \frac{(3\omega_{z0}\omega_x - \dot{\omega_y})\cos\gamma - (3\omega_{z0}\omega_y + \dot{\omega_x})\sin\gamma}{\omega_x \sin\gamma + \omega_y \cos\gamma}, \\
B(\gamma,t) &= \frac{2(\omega_y \sin\gamma - \omega_x \cos\gamma)}{\omega_x \sin\gamma + \omega_y \cos\gamma}, \\
C(\gamma,t) &= \frac{-\omega_x^3 \cos\gamma \sin^2\gamma + 1/4(\sin\gamma - 3\sin(3\gamma))\omega_x^2\omega_y}{\omega_x \sin\gamma + \omega_y \cos\gamma} \\
&+ \frac{1/2\omega_x \cos\gamma(-2\omega_{z0}^2 + (1 - 3\cos(2\gamma))\omega_y^2)}{\omega_x \sin\gamma + \omega_y \cos\gamma} \\
&+ \frac{(\omega_{z0}^2\omega_y + \omega_y^3 \cos^2\gamma + \omega_{z0}\dot{\omega_x})\sin\gamma + \omega_{z0}\dot{\omega_y}\cos\gamma}{\omega_x \sin\gamma + \omega_y \cos\gamma}.
\end{aligned} \tag{11}$$

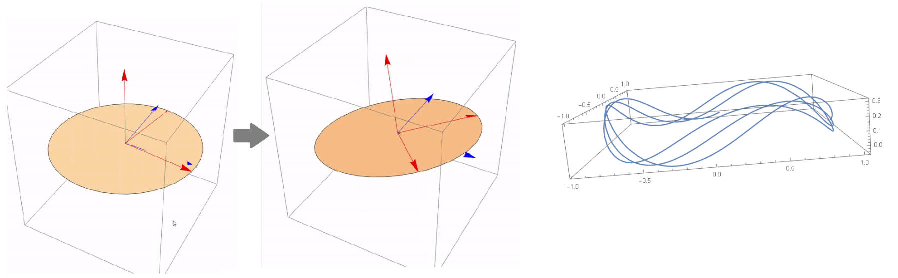
After finding a numerical solution for equation (10) we can easily obtain the solution for β and α.

3. RESULTS AND DISCUSSION

In this section we present the results of our simulations for an imaginary unbiased coin with the values $I_1 = 5 \times 10^{-7}$, $I_3 = I_1/2$ and $R = 10^{-2}$ expressed in SI units. In order to cover a real toss, we take the interval of 3 m/s to 5 m/s for v_0 (which approximately corresponds to duration of motion in the interval 0.6 s to 1 s). The nonlinear second order differential equation (10) is integrated using the program Mathematica. A numerical integration of the system of differential equations (5) is also done with Mathematica and the results of the two ways of numerical integration coincide in the limit of desired accuracy.

3.1. NO COMPLETE ROTATION ABOUT AN AXIS SITUATED IN THE HORIZONTAL PLANE

In this case the coin can not completely rotate about an axis situated in the horizontal plane. This happens because the numerical values of ω_x and ω_y from equations (2) are oscillatory. The coin does not complete a full rotation about an axis situated in the horizontal plane during the flight. In terms of the Euler angles, the value of β does not exceed 180 degrees.



a) b)

Fig. 2 – a) Initial and at a time position of the coin; b) the trace of a vector centered in the plane of the coin, pointing radially, on the surface of a sphere with the center in the center of the coin during the flight. The values of the initial angular velocities are as follows: $\omega_{x0} = \omega_{y0} = 1, \omega_{z0} = 2.5$ expressed in SI units.

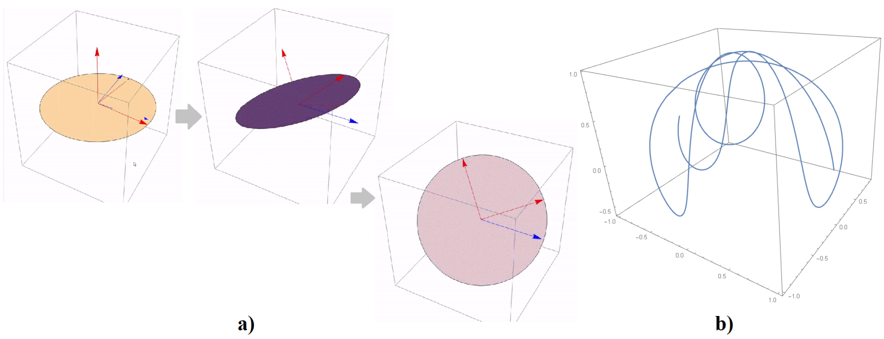
We compute cosine of the dihedral angle as the scalar product between the normal to the surface of the coin, Σ, and the normal to the horizontal plane, H (see Ref. [1]):

$$\cos\theta = \boldsymbol{n}(\Sigma) \cdot \boldsymbol{n}(H).$$

Considering that the coin starts from the horizontal position with head up, we have: if $\cos\theta > 0$ then the head is up; if $\cos\theta < 0$ then the tail is up.

3.2. COMPLETE ROTATION ABOUT AN AXIS SITUATED IN THE HORIZONTAL PLANE

In this case the coin rotates about an axis situated in the horizontal plane because the numerical values of ω_x and ω_y oscillate in a high enough interval.



a) b)

Fig. 3 – a) Initial and at different times position of the coin; b) the trace of a vector centered in the plane of the coin, pointing radially, on the surface of a sphere with the center in the center of the coin during the flight. The values of the initial angular velocities are as follows:$\omega_{x0} = \omega_{y0} = 2, \omega_{z0} = 0.5$ expressed in SI units.

We are interested in the outcome of real tosses. The simulation uses the fact that the outcome of the toss depends on two parameters: the velocity of the CM (or equivalently, duration of motion) and the angular velocity. In the first method (A) we choose a domain of values for the initial parameters (that we name the configuration space) and calculate the outcome of a toss with some set of initial parameters. After having checked all the values in the configuration space we calculate the probability as the ratio between the number of favorable case configurations (head up, for example) and the total number of configurations. In the second method (B) we choose a boundary for the initial parameters and let them vary randomly inside those boundaries for a number of steps.

For the fair toss of an unbiased coin, as expected, an unbiased value of 50% is found in Ref. [1]. However the "surprise" arises in the case we discuss where the probability of the coin landing either heads or tails is not 50% anymore. This is not an outcome to be surprised of as there is no such thing as "magic" involved in the computations, only the right choosing of the initial values for the angular velocities of the coin.

In figure 4 we show the probability evolution as more values of the initial conditions are parsed. For the case we have simulated the probability of the coin landing on the same side as it had been launched came out to about 66%. We have obtained this value for the probability by choosing a space of 250000 configurations for method A and by computing the outcome of the same number of coin throws with a random set of initial conditions for method B.

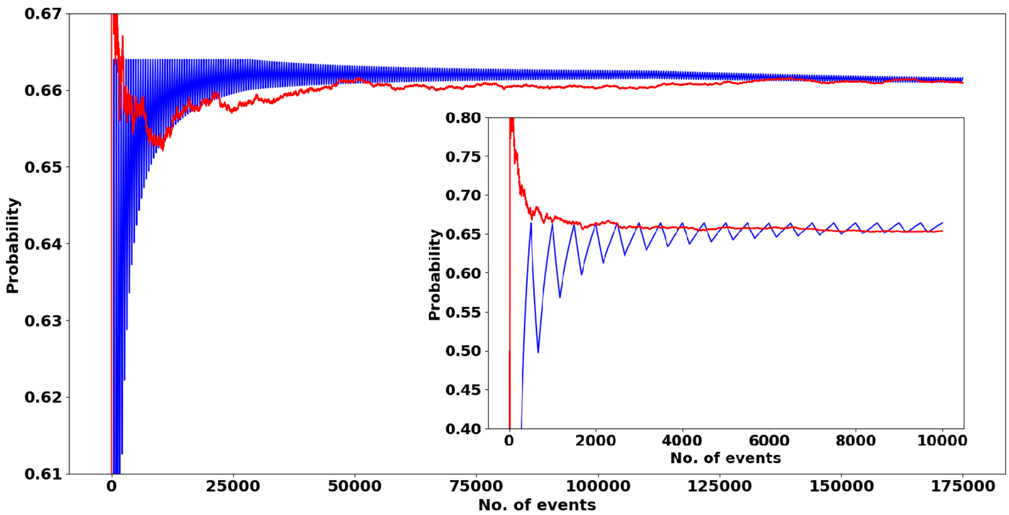


Fig. 4 – Probability computed by method A (blue) and method B (red).

We checked the correctness of our numerical calculations by several way as follows:

a) The angular momentum in the BF, $\boldsymbol{S}$, has the numerically computed magnitude identical (in the limit of precision calculus) with the analytic value $|\boldsymbol{S}| = |I(\omega_x,\omega_y,\omega_z)^t| = \sqrt{I_1^2(\omega_{x0}^2+\omega_{y0}^2)+I_3^2\omega_{z0}^2}$;

b) The kinetic energy in the BF, $\boldsymbol{\omega}\cdot\boldsymbol{S}/2$, has the numerically computed magnitude identical (in the limit of precision calculus) with the analytic value $[I_1(\omega_{x0}^2+\omega_{y0}^2)+I_3\omega_{z0}^2]/2$;

c) Diaconis' theorem [2], $\boldsymbol{n}(t)\cdot\boldsymbol{s}(t) = \boldsymbol{N}(t)\cdot\boldsymbol{S}(t) = constant$ (where $\boldsymbol{N}(t) = \boldsymbol{A}(t)\boldsymbol{n}(t)$ and $\boldsymbol{S}(t) = \boldsymbol{A}(t)\boldsymbol{s}(t)$ are the normal to the coin and the angular momentum in the TF, respectively, and $\boldsymbol{A}(t)$ is the Euler rotation matrix at moment t (see Ref. [1])), is checked;

d) For the no complete rotation about an axis in the horizontal plane (cheated tossing) the head up event probability obtained is equal to 1.

4. CONCLUSIONS

We have modeled the coin toss by using a deterministic torque-free rigid body dynamics for the case where the coin has initial precession velocity. The probability is biased and a skilled person able to launch the coin with initial non-zero angular velocities for all three spatial directions can influence the tossing outcome at will. The numerical code under Mathematica for computing the orientation of the tossed coin and the outcome probability may be asked to the corresponding author.

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