SOLITON, MULTIPLE-LUMP, AND HYBRID SOLUTIONS FOR A (3+1)-DIMENSIONAL GENERALIZED KONOPELCHENKO-DUBROVSKY-KAUP-KUPERSHMIDT EQUATION IN PLASMA PHYSICS, FLUID MECHANICS, AND OCEAN DYNAMICS

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Abstract. In this paper, we investigate a (3+1)-dimensional generalized Konopelchenko-Dubrovsky-Kaup-Kupershmidt equation in plasma physics, fluid mechanics, and ocean dynamics. Based on the existing bilinear form, we construct the N-soliton solutions, where N is a positive integer. Besides, we obtain the M-lump solutions *via* the long-wave limit to the N-soliton solutions, where M is a positive integer. Dark one-lump solutions are derived. Moreover, we take $M = 2$ and $M = 3$ to derive the two- and three-lump solutions, respectively. We find that the shapes and amplitudes of the dark one-, two-, and three-lump waves remain unchanged during propagation. Finally, three types of hybrid solutions are discussed, namely, the one-lump wave and one-soliton wave, the one-lump wave and two-soliton waves, and the two-lump waves and one-soliton wave.

Key words: Plasma physics; Fluid mechanics; Ocean dynamics; Soliton solutions; Multiple-lump solutions; Hybrid solutions.

1. INTRODUCTION

The higher-order nonlinear evolution equations have caught people's attention for their many applications in plasma physics, fluid mechanics, ocean dynamics, etc. [1–14]. Solitons, lumps, and other types of nonlinear waves have been investigated in diverse nonlinear evolution equations [15, 16]. Solitons, which can keep their shapes and amplitudes unchanged during propagation, have been seen as localized waves in certain directions [17]. In comparison with the solitons, the lump waves have been known as localized waves in all directions of space [18]. The recent articles [2, 19] have investigated the following (3+1)-dimensional generalized Konopelchenko-Dubrovsky-Kaup-Kupershmidt equation that describes some nonlinear phenomena in plasma physics, fluid mechanics, and ocean dynamics:

$$v_t + a_1 v_{xxx} + a_2 v v_x + a_3 v_{xxxxx} + a_4 \int v_{yy} \mathrm{dx} + a_5 v_{xxy} + a_6 (v_x \int v_y \mathrm{dx} + v v_y) + a_7 (v_x v_{xx} + v v_{xxx}) + a_8 v^2 v_x + a_9 v_z = 0, \quad (1)$$

where v is a function of the variables x, y, z, and t, the subscripts stand for the partial derivatives, and a_ε's (ε=1, 2, ..., 9) are real constants. Special cases for Eq. (1) have been given: (1) a (2+1)-dimensional generalization of the Korteweg-de Vries equation, which can be applied in the study of interaction of a Riemann wave propagating along the y axis and a long wave propagating along the x axis, with v as the amplitude or elevation of the Riemann wave [20]; (2) a (3+1)-dimensional generalized B-type Kadomtsev-Petviashvili equation for the weakly dispersive waves in a fluid, with v as the amplitude of the weakly dispersive wave in a fluid [21]; (3) a (2+1)-dimensional B-type Kadomtsev-Petviashvili equation for the shallow water waves in a fluid or electrostatic wave potential in a plasma [22]; (4) a (2+1)-dimensional generalized Konopelchenko-Dubrovsky-Kaup-Kupershmidt equation, which describes certain phenomena in plasma physics, ocean dynamics, and fluid mechanics, with v as the amplitude of the relevant wave [23].

By virtue of the transformation

$$v = 12a_1 a_2^{-1}(\ln h)_{xx}, \tag{2}$$

under the coefficient constraints

$$2a_1 a_2^{-1} = 5a_3 a_7^{-1} = a_5 a_6^{-1} = a_7 a_8^{-1} = -10a_9(5a_2 + 2a_7)^{-1}, \tag{3}$$

in Ref. [2] it has been obtained the following bilinear form:

$$(D_x D_t + a_1 D_x^4 + a_3 D_x^6 + a_4 D_y^2 + a_5 D_x^3 D_y + a_9 D_x D_z) h \cdot h = 0, \tag{4}$$

where the operator D is defined as [24]

$$D_x^{\phi_1} D_y^{\phi_2} D_z^{\phi_3} D_t^{\phi_4} (\Omega_1 \cdot F_1) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^{\phi_1} \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^{\phi_2} \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z'}\right)^{\phi_3}$$
$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^{\phi_4} \Omega_1(x,y,z,t) \cdot F_1(x',y',z',t') \,|_{x=x',y=y',z=z',t=t'},$$

with x', y', z', t' being the independent variables, F_1 being a real differentiable function of x', y', z', and t', Ω_1 being a real differentiable function of x, y, z, and t, and ϕ_1, ϕ_2, ϕ_3, and ϕ_4 being non-negative integers. Bilinear form, breather, bright one-lump, one-soliton, and two-soliton solutions for Eq. (1) have been derived using the Hirota method [2]. We mention that in Ref. [19] has been reproduced the Bilinear Form (4) for Eq. (1) using the Bell polynomials and have been constructed some auto-Bäcklund transformations with the symbolic computation method.

To the best of our knowledge, N-soliton, multiple-lump, and hybrid solutions of Eq. (1) have not been obtained, where N is a positive integer. In Sec. 2, based on Bilinear Form (4), we will derive the N-soliton solutions for Eq. (1). In Sec. 3, we will obtain the M-lump solutions of Eq. (1) using the long-wave limit method, where M is a positive integer. The dark one-lump solutions of Eq. (1) will be obtained when $M = 1$. Besides, taking $M = 2$ and $M = 3$, we will derive the two- and three-lump solutions of Eq. (1), respectively. In Sec. 4, three types of hybrid solutions of Eq. (1) will be discussed graphically, which contain the one-lump wave and one-soliton wave, the one-lump wave and two-soliton waves, and the two-lump waves and one-soliton wave hybrid solutions. In Sec. 5 we will give our conclusions.

2. N-SOLITON SOLUTIONS FOR EQ. (1)

In this Section, we investigate the N-soliton solutions of Eq. (1). Substituting the power series expansion of h with regard to a small parameter ϱ,

$$h = 1 + \varrho h_1 + \varrho^2 h_2 + \varrho^3 h_3 + \cdots + \varrho^N h_N,$$

into the Bilinear Form (4) and equating the coefficients of the same power of ϱ to zero, we can derive the soliton solutions of Eq. (1), where h_Ξ's (Ξ = 1, 2, 3, $\cdots$) are real functions of x, y, z, and t. Then the N-soliton solutions of Eq. (1) can be written as

$$v = 12 a_1 a_2^{-1} (\ln h)_{xx}, \tag{5}$$

$$h = h_N = \sum_{\gamma=0,1} \exp\left(\sum_{i=1}^{N} \gamma_i \alpha_i + \sum_{i<j}^{N} \gamma_i \gamma_j B_{ij} \right), \tag{6}$$

where

$$\alpha_i = b_i(x + c_i y + d_i z + e_i t) + \alpha_i^0, \quad e^{B_{ij}} = \frac{B_1}{B_2},$$
$$e_i = -a_1 b_i^2 - a_3 b_i^4 - a_5 b_i^2 c_i - a_4 c_i^2 - a_9 d_i,$$
$$B_1 = 3a_1(b_i - b_j)^2 + 5a_3(b_i - b_j)^2(b_i^2 - b_i b_j + b_j^2) + 2a_5 b_i^2 c_i - 3a_5 b_i b_j c_i$$
$$+ a_5 b_j^2 c_i - a_4 c_i^2 + a_5 b_i^2 c_j - 3a_5 b_i b_j c_j + 2a_5 b_j^2 c_j + 2a_4 c_i c_j - a_4 c_j^2,$$
$$B_2 = 3a_1(b_i + b_j)^2 + 5a_3(b_i + b_j)^2(b_i^2 + b_i b_j + b_j^2) + 2a_5 b_i^2 c_i + 3a_5 b_i b_j c_i$$
$$+ a_5 b_j^2 c_i - a_4 c_i^2 + a_5 b_i^2 c_j + 3a_5 b_i b_j c_j + 2a_5 b_j^2 c_j + 2a_4 c_i c_j - a_4 c_j^2,$$

with α_i, b_i, c_i, d_i, and α_i^0 being constants, $i = 1, 2, \ldots, N$. The notation γ= 0, 1 represents the summation throughout the feasible compositions of γ_1= 0, 1, γ_2= 0, 1, ..., γ_N= 0, 1.

3. M-LUMP SOLUTIONS OF EQ. (1)

The long-wave limit method can produce the M-lump solutions from N-soliton solutions. Therefore, taking the long-wave limit to the N-soliton solutions of Eq. (1), we will obtain the M-lump solutions of Eq. (1). Setting $\alpha_i^0 = i\pi$, we obtain

$$h_N = \sum_{\gamma=0,1} \prod_{i=1}^{N} (-1)^{\gamma_i} \exp(\gamma_i \eta_i) \prod_{i<j}^{N} \exp(\gamma_i \gamma_j B_{ij}),$$

where $\eta_i = b_i[x + c_i y + d_i z + (-a_1 b_i^2 - a_3 b_i^4 - a_5 b_i^2 c_i - a_4 c_i^2 - a_9 d_i)t]$. We derive the following form when we take $b_i \to 0$:

$$h = h_N = \sum_{\gamma=0,1} \prod_{i=1}^{N} (-1)^{\gamma_i} (1 + \gamma_i b_i \kappa_i) \prod_{i<j}^{N} \exp(1 + \gamma_i \gamma_j b_i b_j F_{ij}) + O(b^{N+1}). \quad (7)$$

The rational solutions can be derived when we substitute the Expression (7) into the Transformation (2). Then we find that $v = 12a_1 a_2^{-1} (\ln h_N / \prod_{i=1}^{N} b_i)_{xx}$ are also solutions of Eq. (1). We omit the constant factor $\prod_{i=1}^{N} b_i$ of h_N in Expression (7) and still mark it as h_N, because we want to write out the solutions explicitly. Therefore, we can write the reduced h_N as

$$h_N = \prod_{i=1}^{N} \kappa_i + \frac{1}{2} \sum_{i,j}^{N} F_{ij} \prod_{\gamma_1 \neq i,j}^{N} \kappa_{\gamma_1} + \cdots + \frac{1}{M! 2^M} \sum_{i,j,\ldots,m,n}^{N} \overbrace{F_{ij} F_{kl} \cdots F_{mn}}^{M} \times \prod_{\gamma_2 \neq i,j,k,l,\ldots,m,n}^{N} \kappa_{\gamma_2} + \cdots, \quad (8)$$

where $\kappa_i = x + c_i y + d_i z + (-a_4 c_i^2 - a_9 d_i)t$ and $F_{ij} = \frac{6[2a_1 + a_5(c_i + c_j)]}{a_4 (c_i - c_j)^2}$. $\sum_{i,j,\ldots,m,n}^{N}$ denotes the summation over all feasible compositions of $i, j, \ldots, m, n$, which have taken different values from $1, 2, \ldots, N$. We derive the M-lump solutions of Eq. (1) when we take $F_{ij} > 0$ and $c_{M+i} = c_i^*$ in h_N, $N = 2M$, where "*" represents the complex conjugation. Because the bright one-lump solutions of Eq. (1) have been presented in Ref. [2], we plan to obtain below the dark one-lump solutions.

3.1. DARK ONE-LUMP SOLUTIONS OF EQ. (1)

In order to obtain the dark one-lump solutions of Eq. (1), we take $M = 1$, $N = 2$ and $\alpha_1^0 = \alpha_2^0 = i\pi$. According to the Expression (6), we obtain

$$h_2 = 1 - \exp[\eta_1] - \exp[\eta_2] + \exp[\eta_1 + \eta_1 + B_{12}], \quad (9)$$

where $\eta_i = b_i[x + c_i y + d_i z + (-a_1 b_i^2 - a_3 b_i^4 - a_5 b_i^2 c_i - a_4 c_i^2 - a_9 d_i)t]$, $i = 1, 2$. When we take $b_i \to 0$, $c_1 = c_2^* = g_1 + i g_2$, $d_1 = d_2^* = l_1 + i l_2$ and omit the factor $b_1 b_2$ in Expression (9), we derive

$$h_2 = \kappa_1 \kappa_2 + F_{12}, \quad (10)$$

where $\kappa_i = x + c_i y + d_i z + (-a_4 c_i^2 - a_9 d_i)t$ and $F_{12} = \frac{6[2a_1 + a_5(c_1+c_2)]}{a_4(c_1-c_2)^2}$. Substituting the Expression (10) into the Transformation (2), we obtain the dark one-lump solutions as

$$v_2 = 12 a_1 a_2^{-1} (\ln h_2)_{xx}. \tag{11}$$

Figure 1 displays the dark one-lump wave *via* Solutions (11). We see that the dark one-lump wave has two crests and one trough. During propagation, the amplitude and shape of the dark one-lump wave remain unchanged.

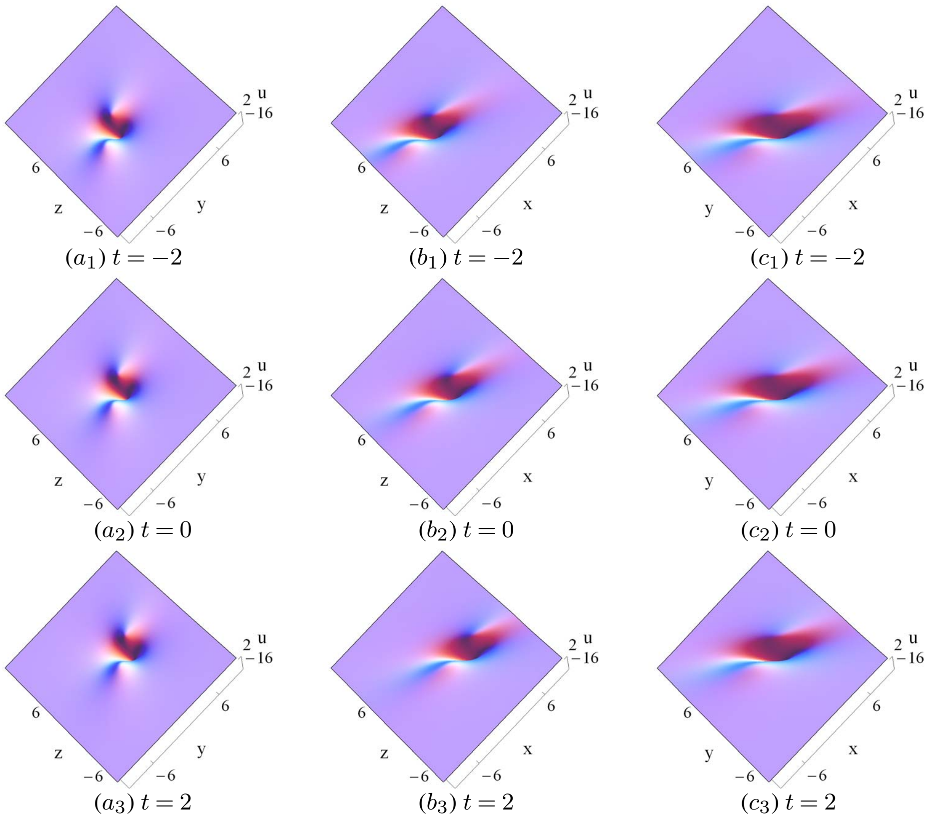


Fig. 1. Dark one-lump wave *via* Solutions (11) with $g_1 = l_1 = a_2 = a_4 = a_5 = a_9 = 1$, $g_2 = -1$, $l_2 = \frac{3}{2}$, $a_1 = -2$ and (a) $x = 0$; (b) $y = 0$; (c) $z = 0$.

3.2. TWO-LUMP SOLUTIONS OF EQ. (1)

For the purpose of constructing the two-lump solutions of Eq. (1), we substitute $M = 2$, $N = 4$ and $\alpha_j^0 = i\pi$ $(j = 1, 2, \ldots, 4)$ into Expression (8), then the reduced h_4

can be written as

$$h_4 = \kappa_1\kappa_2\kappa_3\kappa_4 + F_{12}\kappa_3\kappa_4 + F_{13}\kappa_2\kappa_4 + F_{14}\kappa_2\kappa_3 + F_{23}\kappa_1\kappa_4 + F_{24}\kappa_1\kappa_3 + F_{34}\kappa_1\kappa_2 + F_{12}F_{34} + F_{13}F_{24} + F_{14}F_{23}, \tag{12}$$

where $\kappa_i = x + c_i y + d_i z + (-a_4 c_i^2 - a_9 d_i)t$ and $F_{ij} = \frac{6[2a_1 + a_5(c_i + c_j)]}{a_4(c_i - c_j)^2}$, $1 \le i < j \le 4$. Taking c_1=$c_3^* = g_1 + ig_2$, c_2=$c_4^* = g_3 + ig_4$, $d_1 = d_2 = d_3 = d_4 = d$ and substituting the Expression (12) into the Transformation (2), we obtain the two-lump solutions of Eq. (1) as

$$v_4 = 12a_1 a_2^{-1}(\ln h_4)_{xx}. \tag{13}$$

Figure 2 shows the two-lump waves *via* Solutions (13). We observe that the two-lump waves are composed of two lumps, each of them has one crest and two troughs. The two-lump waves keep their amplitudes and shapes unchanged, as t goes on. In addition, we find that there is a one-strip (that is, a one-line) soliton wave on the $x-z$ plane. The amplitude of the line soliton decreases when $t = 0$.

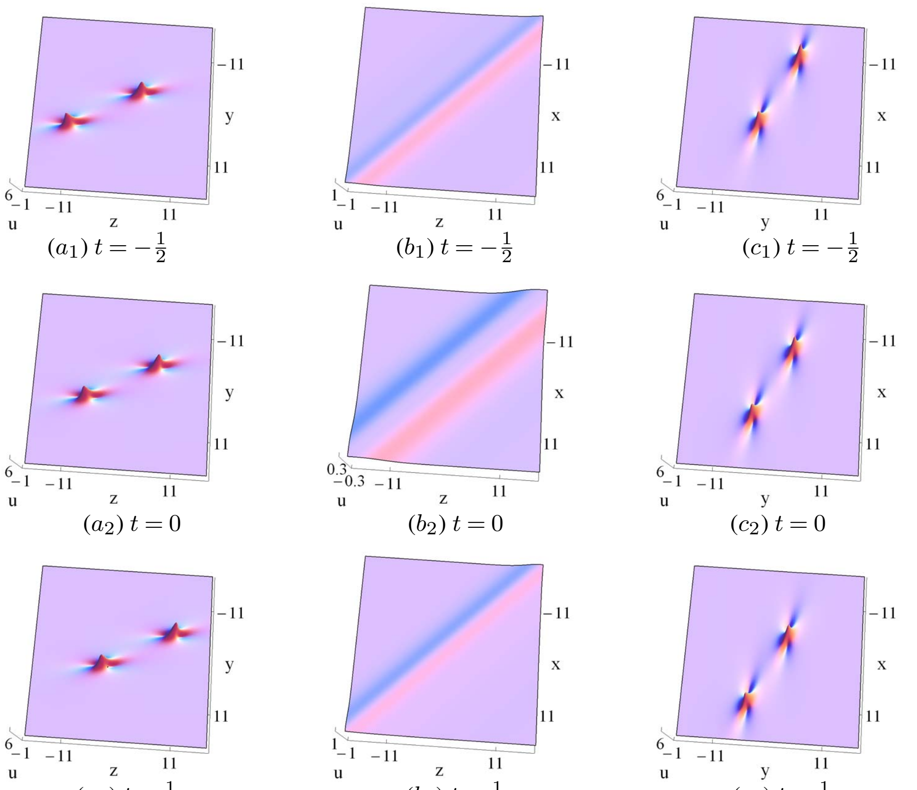


Fig. 2. Two-lump waves *via* Solutions (13) with $g_1 = g_3 = d = a_1 = a_5 = a_9 = 1$, $g_2 = \frac{5}{2}$, $g_4 = \frac{9}{5}$, $a_2 = 4$, $a_4 = -1$ and (a) $x = 0$; (b) $y = 0$; (c) $z = 0$.

3.3. THREE-LUMP SOLUTIONS OF EQ. (1)

When we take $M=3$, $N=6$, and $\alpha_j^0=0$, we obtain the three-lump solutions of Eq. (1). Based on Expression (8), we get the reduced h_6 as

$$\begin{aligned} h_6 = {} & \kappa_1\kappa_2\kappa_3\kappa_4\kappa_5\kappa_6 + F_{12}\kappa_3\kappa_4\kappa_5\kappa_6 + F_{13}\kappa_2\kappa_4\kappa_5\kappa_6 + F_{14}\kappa_2\kappa_3\kappa_5\kappa_6 \\ & + F_{15}\kappa_2\kappa_3\kappa_4\kappa_6 + F_{16}\kappa_2\kappa_3\kappa_4\kappa_5 + F_{23}\kappa_1\kappa_4\kappa_5\kappa_6 + F_{24}\kappa_1\kappa_3\kappa_5\kappa_6 \\ & + F_{25}\kappa_1\kappa_3\kappa_4\kappa_6 + F_{26}\kappa_1\kappa_3\kappa_4\kappa_5 + F_{34}\kappa_1\kappa_2\kappa_5\kappa_6 + F_{35}\kappa_1\kappa_2\kappa_4\kappa_6 \\ & + F_{36}\kappa_1\kappa_2\kappa_4\kappa_5 + F_{45}\kappa_1\kappa_2\kappa_3\kappa_6 + F_{46}\kappa_1\kappa_2\kappa_3\kappa_5 + F_{56}\kappa_1\kappa_2\kappa_3\kappa_4 \\ & + F_{12}F_{34}\kappa_5\kappa_6 + F_{12}F_{35}\kappa_4\kappa_6 + F_{12}F_{36}\kappa_4\kappa_5 + F_{12}F_{45}\kappa_3\kappa_6 \\ & + F_{12}F_{46}\kappa_3\kappa_5 + F_{12}F_{56}\kappa_3\kappa_4 + F_{13}F_{24}\kappa_5\kappa_6 + F_{13}F_{25}\kappa_4\kappa_6 \\ & + F_{13}F_{26}\kappa_4\kappa_5 + F_{13}F_{45}\kappa_2\kappa_6 + F_{13}F_{46}\kappa_2\kappa_5 + F_{13}F_{56}\kappa_2\kappa_4 \\ & + F_{14}F_{23}\kappa_5\kappa_6 + F_{14}F_{25}\kappa_3\kappa_6 + F_{14}F_{26}\kappa_3\kappa_5 + F_{14}F_{35}\kappa_2\kappa_6 \\ & + F_{14}F_{36}\kappa_2\kappa_5 + F_{14}F_{56}\kappa_2\kappa_3 + F_{15}F_{23}\kappa_4\kappa_6 + F_{15}F_{24}\kappa_3\kappa_6 \\ & + F_{15}F_{26}\kappa_3\kappa_4 + F_{15}F_{34}\kappa_2\kappa_6 + F_{15}F_{36}\kappa_2\kappa_4 + F_{15}F_{46}\kappa_2\kappa_3 \\ & + F_{16}F_{23}\kappa_4\kappa_5 + F_{16}F_{24}\kappa_3\kappa_5 + F_{16}F_{25}\kappa_3\kappa_4 + F_{16}F_{34}\kappa_2\kappa_5 \\ & + F_{16}F_{35}\kappa_2\kappa_4 + F_{16}F_{45}\kappa_2\kappa_3 + F_{23}F_{45}\kappa_1\kappa_6 + F_{23}F_{46}\kappa_1\kappa_5 \\ & + F_{23}F_{56}\kappa_1\kappa_4 + F_{24}F_{35}\kappa_1\kappa_6 + F_{24}F_{36}\kappa_1\kappa_5 + F_{24}F_{56}\kappa_1\kappa_3 \\ & + F_{25}F_{34}\kappa_1\kappa_6 + F_{25}F_{36}\kappa_1\kappa_4 + F_{25}F_{46}\kappa_1\kappa_3 + F_{26}F_{34}\kappa_1\kappa_5 \\ & + F_{26}F_{35}\kappa_1\kappa_4 + F_{26}F_{45}\kappa_1\kappa_3 + F_{34}F_{56}\kappa_1\kappa_2 + F_{35}F_{46}\kappa_1\kappa_2 \\ & + F_{36}F_{45}\kappa_1\kappa_2 + F_{12}F_{34}F_{56} + F_{12}F_{35}F_{46} + F_{12}F_{36}F_{45} \\ & + F_{13}F_{24}F_{56} + F_{13}F_{25}F_{46} + F_{13}F_{26}F_{45} + F_{14}F_{23}F_{56} \\ & + F_{14}F_{25}F_{36} + F_{14}F_{26}F_{35} + F_{15}F_{23}F_{46} + F_{15}F_{24}F_{36} \\ & + F_{15}F_{26}F_{34} + F_{16}F_{23}F_{45} + F_{16}F_{24}F_{35} + F_{16}F_{25}F_{34}, \end{aligned} \tag{14}$$

where $\kappa_i = x + c_i y + d_i z + (-a_4 c_i^2 - a_9 d_i)t$ and $F_{ij} = \frac{6[2a_1 + a_5(c_i+c_j)]}{a_4(c_i-c_j)^2}$, $1 \le i < j \le 6$. In order to get the three-lump solutions of Eq. (1), we take $c_1 = c_4^* = g_1 + ig_2$, $c_2 = c_5^* = g_3 + ig_4$, $c_3 = c_6^* = g_5 + ig_6$, $d_1 = d_2 = d_3 = d_4 = d_5 = d_6 = d$, and substitute the Expression (14) into the Transformation (2). Then we obtain the three-lump solutions of Eq. (1) as

$$v_6 = 12a_1 a_2^{-1}(\ln h_6)_{xx}. \tag{15}$$

Figure 3 shows the three-lump waves *via* Solutions (15). We see that the three-lump waves are composed of three lumps, each of them has one crest and two troughs. As t goes on, the three-lump waves keep their amplitudes and shapes unchanged. Besides, we find that there is a one-strip soliton wave on the $x-z$ plane. The amplitude of the one-strip soliton increases when $t=0$.

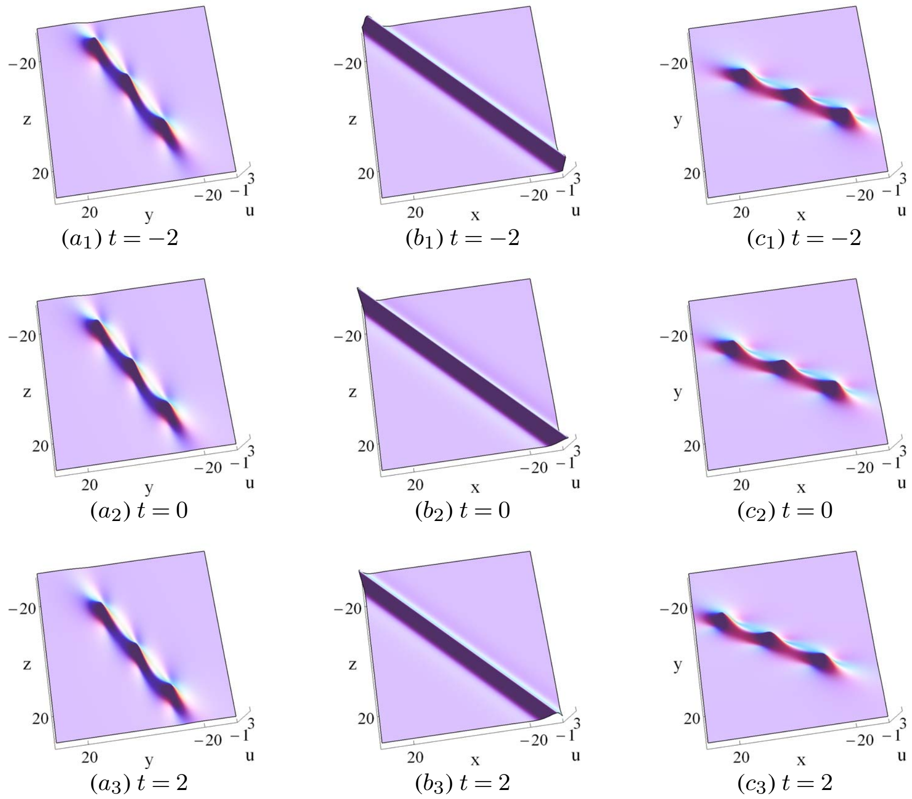


Fig. 3. Three-lump waves *via* Solutions (15) with $g_1=g_2=g_3=g_5=d=a_1=a_2=a_5=a_9=1$, $g_4=2$, $g_6=3$, $a_4=-\frac{1}{5}$ and (a) $x=0$; (b) $y=0$; (c) $z=0$.

4. HYBRID SOLUTIONS OF EQ. (1)

4.1. THE ONE-LUMP WAVE AND ONE-SOLITON WAVE HYBRID SOLUTIONS

In this part, we study the hybrid solutions composed of one-lump wave and one-soliton wave, which can be derived from h_N when we take $N=3$. Letting the parameters $\alpha_1^0=\alpha_2^0=i\pi$, $b_1\to 0$, and $b_2\to 0$, we obtain $h_{3\tau}$ as

$$h_{3\tau}=\kappa_1\kappa_2+F_{12}+(F_{13}F_{23}+F_{13}\kappa_2+F_{23}\kappa_1+F_{12}+\kappa_1\kappa_2)\exp(\alpha_3), \tag{16}$$

where

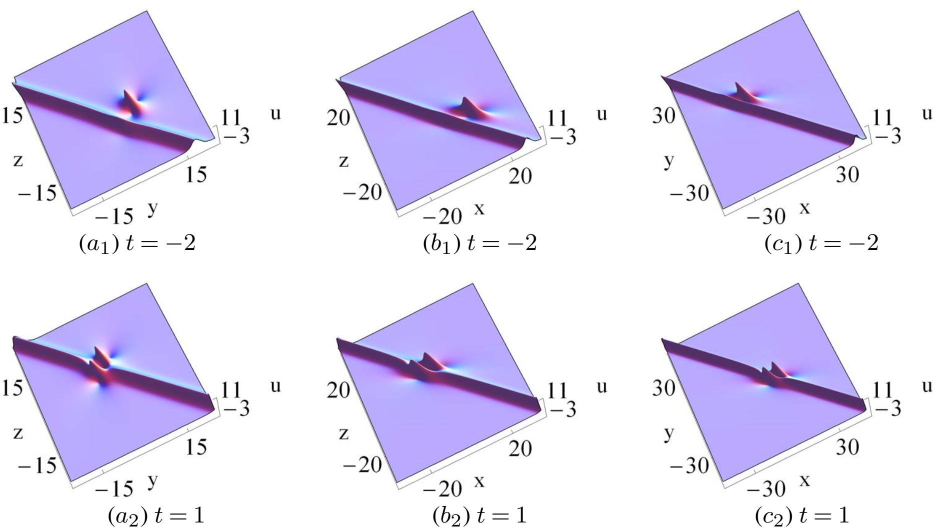
$$\begin{aligned}
&\kappa_1 = x + c_1 y + d_1 z + (-a_4 c_1^2 - a_9 d_1)t,\\
&\kappa_2 = x + c_2 y + d_2 z + (-a_4 c_2^2 - a_9 d_2)t,\\
&\alpha_3 = b_3[x + c_3 y + d_3 z + (-a_1 b_3^2 - a_3 b_3^4 - a_5 b_3^2 c_3 - a_4 c_3^2 - a_9 d_3)t] + \alpha_{30},\\
&c_1 = c_2^* = g_1 + i g_2, \quad d_1 = d_2^* = g_3 + i g_4,
\end{aligned}$$

$$F_{\nu_1\nu_2} = \begin{cases} \frac{6[2a_1 + a_5(c_{\nu_1} + c_{\nu_2})]}{a_4(c_{\nu_1} - c_{\nu_2})^2}, & \nu_2 < 3, \\ -\frac{6[2a_1 + 5a_3 b_{\nu_2}^2 + a_5(c_{\nu_1} + c_{\nu_2})]}{3a_1 b_{\nu_2}^2 + 5a_3 b_{\nu_2}^4 + a_5 b_{\nu_2}^2 c_{\nu_1} - a_4 c_{\nu_1}^2 + 2a_5 b_{\nu_2}^2 c_{\nu_2} + 2a_4 c_{\nu_1} c_{\nu_2} - a_4 c_{\nu_2}^2}, & \nu_2 = 3. \end{cases}$$

Substituting the Expression (16) into the Transformation (2), we obtain the following one-lump wave and one-soliton wave hybrid solutions of Eq. (1):

$$v_{3\tau} = 12 a_1 a_2^{-1} (\ln h_{3\tau})_{xx}. \tag{17}$$

Figure 4 shows the one-lump wave and one-soliton wave hybrid solutions *via* Solutions (17). We see that the one-soliton wave and one-lump wave move forward independently at $t = -2$. When $t = 1$, the one-soliton wave gradually merges with the one-lump wave, and then the one-soliton wave separates from the one-lump wave as t goes on.



$(a_1)\ t = -2$ $(b_1)\ t = -2$ $(c_1)\ t = -2$

$(a_2)\ t = 1$ $(b_2)\ t = 1$ $(c_2)\ t = 1$

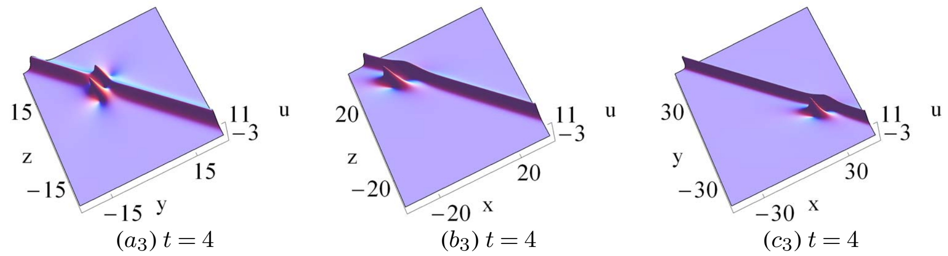


Fig. 4. The one-lump wave and one-soliton wave hybrid solutions *via* Solutions (17) with $g_4 = -1$, $a_1 = 2$, $a_4 = -2$, $g_1 = g_2 = g_3 = a_2 = a_3 = a_5 = a_9 = b_3 = c_3 = d_3 = 1$, $\alpha_{30} = 0$ and (a) $x = 0$; (b) $y = 0$; (c) $z = 0$.

4.2. THE ONE-LUMP WAVE AND TWO-SOLITON WAVES HYBRID SOLUTIONS OF EQ. (1)

Next, we will derive the one-lump wave and two-soliton waves hybrid solutions of Eq. (1). Firstly, we take $N = 4$, $\alpha_1^0 = \alpha_2^0 = i\pi$, and $b_1, b_2 \to 0$, then $h_{4\tau}$ can be obtained as follows:

$$\begin{aligned} h_{4\tau} = {} & \kappa_1\kappa_2 + F_{12} + (\kappa_1\kappa_2 + F_{23}\kappa_1 + F_{13}\kappa_2 + F_{13}F_{23} + F_{12})\exp(\alpha_3) \\ & +(\kappa_1\kappa_2 + F_{24}\kappa_1 + F_{14}\kappa_2 + F_{14}F_{24} + F_{12})\exp(\alpha_4) \\ & +[\kappa_1\kappa_2 + (F_{23} + F_{24})\kappa_1 + (F_{13} + F_{14})\kappa_2 + F_{12} + F_{13}F_{23} \\ & +F_{14}F_{23} + F_{13}F_{24} + F_{14}F_{24}]\exp(B_{34})\exp(\alpha_3 + \alpha_4), \end{aligned} \tag{18}$$

where

$$\begin{aligned} & \kappa_1 = x + c_1 y + d_1 z + (-a_4 c_1^2 - a_9 d_1)t, \\ & \kappa_2 = x + c_2 y + d_2 z + (-a_4 c_2^2 - a_9 d_2)t, \\ & \alpha_3 = b_3[x + c_3 y + d_3 z + (-a_1 b_3^2 - a_3 b_3^4 - a_5 b_3^2 c_3 - a_4 c_3^2 - a_9 d_3)t] + \alpha_{30}, \\ & \alpha_4 = b_4[x + c_4 y + d_4 z + (-a_1 b_4^2 - a_3 b_4^4 - a_5 b_4^2 c_4 - a_4 c_4^2 - a_9 d_4)t] + \alpha_{40}, \\ & \exp(B_{34}) = \frac{B_3}{B_4}, \end{aligned}$$

$$F_{\nu_1\nu_2} = \begin{cases} \dfrac{6[2a_1 + a_5(c_{\nu_1} + c_{\nu_2})]}{a_4(c_{\nu_1} - c_{\nu_2})^2}, & \nu_2 < 3, \\ -\dfrac{6[2a_1 + 5a_3 b_{\nu_2}^2 + a_5(c_{\nu_1} + c_{\nu_2})]}{3a_1 b_{\nu_2}^2 + 5a_3 b_{\nu_2}^4 + a_5 b_{\nu_2}^2 c_{\nu_1} - a_4 c_{\nu_1}^2 + 2a_5 b_{\nu_2}^2 c_{\nu_2} + 2a_4 c_{\nu_1} c_{\nu_2} - a_4 c_{\nu_2}^2}, & \nu_2 \geq 3, \end{cases}$$

$$
\begin{aligned}
B_3 =\ & 3a_1(b_3-b_4)^2+5a_3(b_3-b_4)^2(b_3^2-b_3b_4+b_4^2)+2a_5b_3^2c_3-3a_5b_3b_4c_3 \\
& +a_5b_4^2c_3-a_4c_3^2+a_5b_3^2c_4-3a_5b_3b_4c_4+2a_5b_4^2c_4+2a_4c_3c_4-a_4c_4^2, \\
B_4 =\ & 3a_1(b_3+b_4)^2+5a_3(b_3+b_4)^2(b_3^2+b_3b_4+b_4^2)+2a_5b_3^2c_3+3a_5b_3b_4c_3 \\
& +a_5b_4^2c_3-a_4c_3^2+a_5b_3^2c_4+3a_5b_3b_4c_4+2a_5b_4^2c_4+2a_4c_3c_4-a_4c_4^2.
\end{aligned}
$$

Choosing $c_1 = c_2^* = g_1 + ig_2 = d_1 = d_2^*$ and substituting the Expression (18) into the Transformation (2), we obtain the one-lump wave and two-soliton waves hybrid solutions of Eq. (1) as

$$
v_{4\tau} = 12a_1a_2^{-1}(\ln h_{4\tau})_{xx}. \tag{19}
$$

4.3. THE TWO-LUMP WAVES AND ONE-SOLITON WAVE HYBRID SOLUTIONS OF EQ. (1)

For the purpose of deriving the two-lump waves and one-soliton wave hybrid solutions of Eq. (1), we take the long-wave limit to the five-soliton solutions. The Expression of h_{5h} can be written as

$$
\begin{aligned}
h_{5\tau} =\ & \kappa_1\kappa_2\kappa_3\kappa_4+F_{12}\kappa_3\kappa_4+F_{13}\kappa_2\kappa_4+F_{14}\kappa_2\kappa_3+F_{23}\kappa_1\kappa_4+F_{24}\kappa_1\kappa_3 \\
& +F_{34}\kappa_1\kappa_2+F_{12}F_{34}+F_{13}F_{24}+F_{14}F_{23}+\exp(\alpha_5)(F_{15}F_{25}F_{35}F_{45} \\
& +F_{15}F_{25}F_{35}\kappa_4+F_{15}F_{25}F_{45}\kappa_3+F_{15}F_{35}F_{45}\kappa_2+F_{25}F_{35}F_{45}\kappa_1 \\
& +F_{15}F_{25}\kappa_3\kappa_4+F_{15}F_{35}\kappa_2\kappa_4+F_{15}F_{45}\kappa_2\kappa_3+F_{25}F_{35}\kappa_1\kappa_4 \\
& +F_{25}F_{45}\kappa_1\kappa_3+F_{35}F_{45}\kappa_1\kappa_2+F_{15}\kappa_2\kappa_3\kappa_4+F_{25}\kappa_1\kappa_3\kappa_4 \\
& +F_{35}\kappa_1\kappa_2\kappa_4+F_{45}\kappa_1\kappa_2\kappa_3+\kappa_1\kappa_2\kappa_3\kappa_4+F_{12}F_{35}F_{45}+F_{12}F_{35}\kappa_4 \\
& +F_{12}F_{45}\kappa_3+F_{12}\kappa_3\kappa_4+F_{13}F_{25}F_{45}+F_{13}F_{25}\kappa_4+F_{13}F_{45}\kappa_2 \\
& +F_{13}\kappa_2\kappa_4+F_{14}F_{25}F_{35}+F_{14}F_{25}\kappa_3+F_{14}F_{35}\kappa_2+F_{14}\kappa_2\kappa_3 \\
& +F_{15}F_{23}F_{45}+F_{15}F_{23}\kappa_4+F_{15}F_{24}F_{35}+F_{15}F_{24}\kappa_3+F_{15}F_{25}F_{34} \\
& +F_{15}F_{34}\kappa_2+F_{23}F_{45}\kappa_1+F_{23}\kappa_1\kappa_4+F_{24}F_{35}\kappa_1+F_{24}\kappa_1\kappa_3 \\
& +F_{25}F_{34}\kappa_1+F_{34}\kappa_1\kappa_2+F_{12}F_{34}+F_{13}F_{24}+F_{14}F_{23}),
\end{aligned}
$$

where

$$
\begin{aligned}
& \kappa_i = x+c_iy+d_iz+(-a_4c_i^2-a_9d_i)t, \quad (i=1,2,3,4) \\
& \alpha_5 = b_5[x+c_5y+d_5z+(-a_1b_5^2-a_3b_5^4-a_5b_5^2c_5-a_4c_5^2-a_9d_5)t]+\alpha_{50},
\end{aligned}
$$

$$
F_{\nu_1\nu_2} = \begin{cases} \dfrac{6[2a_1+a_5(c_{\nu_1}+c_{\nu_2})]}{a_4(c_{\nu_1}-c_{\nu_2})^2}, & \nu_2 < 5, \\[2ex] -\dfrac{6[2a_1+5a_3b_{\nu_2}^2+a_5(c_{\nu_1}+c_{\nu_2})]}{3a_1b_{\nu_2}^2+5a_3b_{\nu_2}^4+a_5b_{\nu_2}^2c_{\nu_1}-a_4c_{\nu_1}^2+2a_5b_{\nu_2}^2c_{\nu_2}+2a_4c_{\nu_1}c_{\nu_2}-a_4c_{\nu_2}^2}, & \nu_2 = 5. \end{cases}
$$

Taking $c_1 = c_3^* = d_1 = d_3^*$, $c_2 = c_4^* = d_2 = d_4^*$ and substituting the Expression (20) into the Transformation (2), we obtain the following two-lump waves and one-soliton wave hybrid solutions of Eq. (1):

$$v_{5\tau} = 12a_1a_2^{-1}(\ln h_{5\tau})_{xx}. \tag{20}$$

Figure 5 shows the two-lump waves and one-soliton wave hybrid solutions *via* Solutions (20). We observe that there is a one-strip soliton wave on the $y-z$ plane. On the $x-z$ and $x-y$ planes, there are two-lump waves and one-soliton wave when $t=-3$. As t goes on, the two-lump waves approach to the one-soliton wave gradually, and then the two-lump waves interact with the one-soliton wave. When $t=2$, the two-lump waves and one-soliton wave move forward independently.

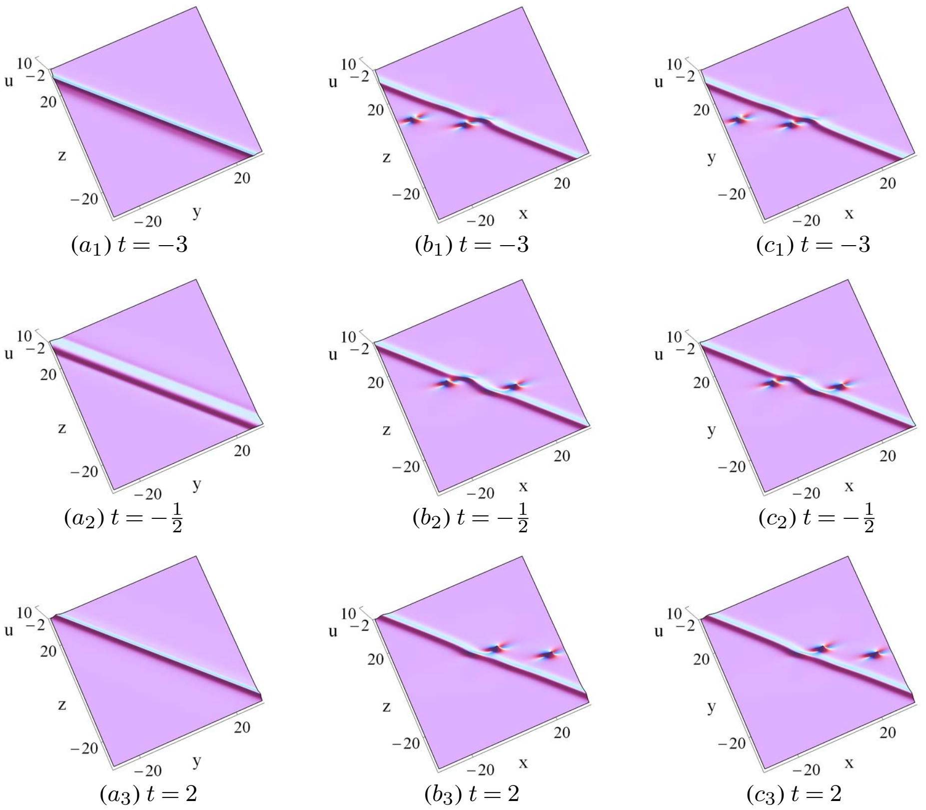


Fig. 5. The two-lump waves and one-soliton wave hybrid solutions *via* Solutions (20) with $g_4=\frac{5}{2}$, $\alpha_{50}=0$, $a_2=5$, $g_1=g_3=d_5=a_5=a_9=b_3=b_5=c_5=1$, $g_2=a_1=b_4=2$, $a_3=a_4=-1$ and (a) $x=0$; (b) $y=0$; (c) $z=0$.

5. CONCLUSIONS

In this paper, we have investigated a (3+1)-dimensional generalized Konopelchenko-Dubrovsky-Kaup-Kupershmidt equation (1), which has applications in plasma physics, fluid mechanics, and ocean dynamics. Based on Bilinear Form (4), we have obtained N-Soliton Solutions (5). Applying the long-wave limit to Solutions (5), we have derived Dark One-Lump Solutions (11), Two-Lump Solutions (13), and Three-Lump Solutions (15). In addition, we have obtained the one-lump wave and one-soliton wave hybrid solutions (17), the one-lump wave and two-soliton waves hybrid solutions (19), and the two-lump waves and one-soliton wave hybrid solutions (20).

Figure 1 has displayed the dark one-lump wave, whose amplitude and shape remain unchanged during propagation and contains two crests and one trough, *via* Solutions (11). In Fig. 2 we have illustrated the two-lump waves, composed of two lumps, *via* Solutions (13). Figure 3 has displayed the three-lump waves according to Solutions (15). In Fig. 4 we have presented the one-lump wave and one-soliton wave hybrid solutions *via* Solutions (17), from which we can observe that the one-soliton wave and one-lump wave move forward independently at $t=-2$, the one-soliton wave gradually merges with the one-lump wave when $t=1$, and then the one-soliton wave separates from the one-lump wave as t goes on. Figure 5 has displayed the two-lump waves and one-soliton wave hybrid solutions according to Solutions (20).

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