ASYMPTOTIC ANALYSIS FOR A DIFFUSION PROBLEM IN THIN FILTERING MATERIALS

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The homogenization of a diffusion problem in a thin periodic heterogeneous composite medium made up of two materials separated by an imperfect interface is studied. This mathematical model might have applications in the analysis of various filtering materials, such as textiles, paper, or biological tissues.

Key words: asymptotic analysis, thin periodic domain, imperfect interface.

1. INTRODUCTION

The goal of this paper is to analyze a diffusion problem in a periodic composite medium made up of two constituents occupying a thin three-dimensional domain of height ε denoted Ω^ε. The two components of the domain Ω^ε, namely Ω_1^ε, supposed to be connected, and, respectively, Ω_2^ε, assumed to be disconnected, are separated by an imperfect interface Γ^ε. Such a structure might be encountered in applications to problems involving filtering materials (textiles, paper, biological tissues) constituted of three thin horizontal layers of total height ε. The top layer and the bottom one are identical (made up of material 1) and the third one in between them, which plays the role of a filter for the whole structure, consists of a periodic mixture of two materials (material 1 and 2), the material 2 corresponding to the subdomain Ω_2^ε being fully included in this layer.

In such a structure, we analyze the macroscopic behavior, when the small parameter ε tends to zero, of the solution $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ of problem (2.2) below. The main features of our problem are the special geometry and the fact that the flux of the solution is discontinuous across the interface separating the two materials and depends in a linear manner on the jump of the solution.

In order to study the asymptotic behavior of problem (2.2), we apply the periodic unfolding method (see [1] and the references therein), with operators adapted to the present geometry (see [2] and [3]). At the limit, we obtain a lower-dimensional

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modified Barenblatt system (see Theorem 4.2 and Remark 4.3), in which the dimension reduction phenomenon appears. The effect of the small height of the domain Ω^ε is reflected in the fact that in the limit the diffusion occurs only in the horizontal bi-dimensional domain ω, where the homogenized problem (4.11) is stated. The derivative with respect to the vertical variable x_3 does not appear in the limit equation, but, still, the solution of the effective problem keeps track of the local vertical variable y_3 via the values of the constant homogenized coefficients (4.13) (see Remark 4.4).

For mathematical studies of diffusion problems in thin periodic media, we refer, for instance, to [4–12] and the references therein. For flow problems in thin porous media, we refer, for example, to [13, 14]. There is a wide literature concerning diffusion problems with jumps in solution (see, for instance, [15–26] and the references therein). The mathematical study of problems with flux jump is more recent (see, for instance, [27–33]). For problems with jumps in thin porous media, very few mathematical results are known by now (see [10] and [34]). The problem studied in [10] involves only jumps in the flux, while the one addressed in [34] is a double-porosity problem which contains jumps in the solution, as well. The problem tackled in the present paper corresponds to the one studied in [31], but here the periodic domain in which the problem is set is thin. This specific geometry brings additional difficulties in getting the macroscopic behavior of the problem. To overcome them, we rely on the use of suitable unfolding operators, which allow us to simultaneously perform homogenization and dimension reduction.

The structure of the paper is as follows: in Section 2, we fix the notation and we set the microscopic problem. In Section 3, we recall the appropriate unfolding operators and their main properties. In Section 4, we state and prove the main homogenization results of this paper.

2. SETTING OF THE PROBLEM

Let ω be a smooth and bounded domain in $\mathbb{R}^2$. The independent variable $x \in \mathbb{R}^3$ is denoted by $x = (x_1, x_2, x_3) = (\bar{x}, x_3)$. We define

$$\Omega^\varepsilon = \omega \times (0, \varepsilon) = \{x = (\bar{x}, x_3) \in \mathbb{R}^3 | \bar{x} \in \omega, 0 < x_3 < \varepsilon\}, \tag{2.1}$$

where $\varepsilon \in (0,1)$ is a small parameter related to the characteristic dimension of our domain and which takes values in a sequence of strictly positive numbers such that $1/\varepsilon \in \mathbb{N}^*$. The domain ω is taken in a such a way that Ω^ε is the union of a finite number (depending on ε) of replicated unit cells $Y = (0,1)^3$, rescaled with ε. We consider that $Y = Y_1 \cup \overline{Y}_2$, where Y_1 and Y_2 are two non-empty disjoint connected open subsets of Y such that $\overline{Y}_2 \subset Y$. We assume that the boundary Γ of Y_2 is Lipschitz continuous. Let $\kappa \in \mathbb{Z}^3$; we set $Y_\alpha^\kappa = \kappa + Y_\alpha$, for $\alpha \in \{1, 2\}$. For each

ε, let $\mathbb{Z}_\varepsilon = \{\kappa \in \mathbb{Z}^3 | \, \varepsilon \overline{Y}_2^\kappa \subset \Omega^\varepsilon\}$; we define $\Omega_2^\varepsilon = \bigcup_{\kappa\in\mathbb{Z}_\varepsilon} (\varepsilon Y_2^\kappa)$ and $\Omega_1^\varepsilon = \Omega \setminus \overline{\Omega}_2^\varepsilon$. The boundary of Ω_2^ε is denoted by Γ^ε and n^ε stands for the unit outward normal to Ω_2^ε. The boundary of the domain Ω^ε consists of three parts: its lateral boundary $\Sigma_D^\varepsilon = \{x \in \mathbb{R}^3 | \, \bar{x} \in \partial\omega, 0 < x_3 < \varepsilon\}$ and, respectively, its top and bottom boundaries $\Sigma_+^{\varepsilon,N} = \{x \in \mathbb{R}^3 | \, \bar{x} \in \omega, x_3 = \varepsilon\}$ and $\Sigma_-^{\varepsilon,N} = \{x \in \mathbb{R}^3 | \, \bar{x} \in \omega, x_3 = 0\}$.

The main goal of our paper is to describe the asymptotic behavior, as $\varepsilon \to 0$, of the solution $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ of the following problem:

$$\begin{cases} -\text{div}\,(A^\varepsilon \nabla u_1^\varepsilon) = f \quad \text{in } \Omega_1^\varepsilon, \\ -\text{div}\,(A^\varepsilon \nabla u_2^\varepsilon) = f \quad \text{in } \Omega_2^\varepsilon, \\ A^\varepsilon \nabla u_1^\varepsilon \cdot n^\varepsilon = \varepsilon h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon) - \varepsilon g^\varepsilon \quad \text{on } \Gamma^\varepsilon, \\ A^\varepsilon \nabla u_2^\varepsilon \cdot n^\varepsilon = \varepsilon h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon) \quad \text{on } \Gamma^\varepsilon, \\ A^\varepsilon \nabla u_1^\varepsilon \cdot \nu_\pm^\varepsilon = \varepsilon k_\pm \quad \text{on } \Sigma_\pm^{\varepsilon,N}, \\ u_1^\varepsilon = 0 \quad \text{on } \Sigma_D^\varepsilon. \end{cases} \tag{2.2}$$

Remark 2.1 *We point out that one has $A^\varepsilon \nabla u_1^\varepsilon \cdot n^\varepsilon - A^\varepsilon \nabla u_2^\varepsilon \cdot n^\varepsilon = -\varepsilon g^\varepsilon$, i.e. the flux of the solution of the above problem is discontinuous across Γ^ε.*

For the data involved in our problem, we make the following assumptions:

(A1) Let $\lambda, \mu \in \mathbb{R}$, with $0 < \lambda \le \mu$ and denote by $\mathcal{M}(\lambda, \mu, Y)$ the set of all the matrices $A = (a_{ij}) \in (L^\infty(Y))^{3\times 3}$ such that for any $\xi \in \mathbb{R}^3$, $\lambda|\xi|^2 \le (A(y)\xi, \xi) \le \mu|\xi|^2$, a.e. in Y. Given a symmetric matrix $A \in \mathcal{M}(\lambda, \mu, Y)$ which is 1–periodic in its first two variables y_1 and y_2, we put

$$A^\varepsilon(x) = A^\varepsilon(\bar{x}, x_3) = A\left(\frac{\bar{x}}{\varepsilon}, \frac{x_3}{\varepsilon}\right) = A\left(\frac{x}{\varepsilon}\right) \text{ a.e. in } \Omega^\varepsilon.$$

(A2) The function $f \in L^2(\omega)$ is given.

(A3) Let $h \in L^\infty(\Gamma)$ be a function which is 1–periodic in its first two variables y_1 and y_2 and such that there exists $h_0 \in \mathbb{R}$ with $0 < h_0 < h(y)$ a.e. on Γ. We define

$$h^\varepsilon(x) = h\left(\frac{x}{\varepsilon}\right) \quad \text{a.e. on } \Gamma^\varepsilon.$$

(A4) Let $g \in L^2(\Gamma)$ be a function which is 1–periodic in its first two variables y_1 and y_2. We set

$$g^\varepsilon(x) = g\left(\frac{x}{\varepsilon}\right) \text{ a.e. on } \Gamma^\varepsilon$$

and we assume that $\mathcal{M}_\Gamma(g) \neq 0$, where $\mathcal{M}_\Gamma(g)$ is the mean value of g on Γ.

(A5) The functions $k_+ \in L^2(\omega)$ and $k_- \in L^2(\omega)$ are given.

Let us consider, for every $\varepsilon \in (0,1)$, the Hilbert space $H^\varepsilon = V^\varepsilon \times H^1(\Omega_2^\varepsilon)$. The space $V^\varepsilon = \{v \in H^1(\Omega_1^\varepsilon) | \, v = 0 \text{ on } \Sigma_D^\varepsilon\}$ will be endowed with the norm $\|v\|_{V^\varepsilon} = \|\nabla v\|_{L^2(\Omega_1^\varepsilon)}$, for any $v \in V^\varepsilon$, and the space $H^1(\Omega_2^\varepsilon)$ will be considered with the standard norm. On the space H^ε, we consider, for $u = (u_1, u_2)$, $v = (v_1, v_2) \in H^\varepsilon$, the scalar product

$$(u,v)_{H^\varepsilon} = \int_{\Omega_1^\varepsilon} \nabla u_1 \cdot \nabla v_1 \,\mathrm{d}x + \int_{\Omega_2^\varepsilon} \nabla u_2 \cdot \nabla v_2 \,\mathrm{d}x + \varepsilon \int_{\Gamma^\varepsilon} (u_1 - u_2)(v_1 - v_2)\,\mathrm{d}\sigma_x \tag{2.3}$$

and the norm

$$\|v\|^2_{H^\varepsilon} = \|\nabla v_1\|^2_{L^2(\Omega_1^\varepsilon)} + \|\nabla v_2\|^2_{L^2(\Omega_2^\varepsilon)} + \varepsilon \|v_1 - v_2\|^2_{L^2(\Gamma^\varepsilon)}. \tag{2.4}$$

The variational formulation of problem (2.2) reads as follows: find $u^\varepsilon \in H^\varepsilon$ s.t.

$$a(u^\varepsilon, v) = l(v), \quad \forall v \in H^\varepsilon, \tag{2.5}$$

where the bilinear form $a : H^\varepsilon \times H^\varepsilon \to \mathbb{R}$ and the linear form $l : H^\varepsilon \to \mathbb{R}$ are

$$a(u,v) = \int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1 \cdot \nabla v_1 \,\mathrm{d}x + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_2 \cdot \nabla v_2 \,\mathrm{d}x + \varepsilon \int_{\Gamma^\varepsilon} h^\varepsilon (u_1 - u_2)(v_1 - v_2)\,\mathrm{d}\sigma_x$$

and, respectively,

$$l(v) = \int_{\Omega_1^\varepsilon} f v_1 \,\mathrm{d}x + \int_{\Omega_2^\varepsilon} f v_2 \,\mathrm{d}x + \varepsilon \int_{\Gamma^\varepsilon} g^\varepsilon v_1 \,\mathrm{d}\sigma_x +$$

$$\varepsilon \int_{\Sigma_+^{\varepsilon,N}} k_+ v_1 \,\mathrm{d}\sigma_x^+ + \varepsilon \int_{\Sigma_-^{\varepsilon,N}} k_- v_1 \,\mathrm{d}\sigma_x^-.$$

In what follows, unless otherwise specified, we denote by C a positive constant which is independent of ε and whose value can change from line to line.

In order to prove the well-posedness of problem (2.5), we recall the following useful lemmas (see [34]).

Lemma 2.2 *For every v given in the space H^ε, the following inequalities hold true:*

$$\|v_1\|_{L^2(\Omega_1^\varepsilon)} \le C\|v\|_{H^\varepsilon}, \qquad \|v_2\|_{L^2(\Omega_2^\varepsilon)} \le C\|v\|_{H^\varepsilon}.$$

Lemma 2.3 *Let $v_1 \in V^\varepsilon$. Under the assumption* **(A4)***, the following estimate holds:*

$$I = \left| \varepsilon \int_{\Gamma^\varepsilon} g^\varepsilon(x) v_1(x)\,\mathrm{d}\sigma_x \right| \le \sqrt{\varepsilon} C \|\nabla v_1\|_{L^2(\Omega_1^\varepsilon)}.$$

Theorem 2.4 *For any $\varepsilon \in (0,1)$, the variational problem* (2.5) *has a unique solution $u^\varepsilon \in H^\varepsilon$. Moreover, there exists a constant $C > 0$, independent of ε, such that*

$$\frac{1}{\sqrt{\varepsilon}} \|u^\varepsilon_\alpha\|_{L^2(\Omega^\varepsilon_\alpha)} \le C, \quad \frac{1}{\sqrt{\varepsilon}} \|\nabla u^\varepsilon_\alpha\|_{L^2(\Omega^\varepsilon_\alpha)} \le C, \quad \alpha \in \{1,2\}, \tag{2.6}$$

$$\|u_1^\varepsilon - u_2^\varepsilon\|_{L^2(\Gamma^\varepsilon)} \le C. \tag{2.7}$$

Proof. The proof of the well-posedness of problem (2.5) is done by using the Lax-Milgram theorem for the space H^ε endowed with the norm (2.4). Thanks to the hypotheses ($\mathbf{A1}$) and ($\mathbf{A2}$), it is easy to see that the bilinear form a is coercive and continuous, i.e. $a(v,v) \geq C\|v\|^2_{H^\varepsilon}$, for all $v \in H^\varepsilon$ and $a(u,v) \leq C\|u\|_{H^\varepsilon}\|v\|_{H^\varepsilon}$, for all $u,v \in H^\varepsilon$. Following the same lines as in [34], we can prove that the linear form l is continuous. More precisely, using Lemma 2.3, we have $|l(v)| \leq \sqrt{\varepsilon}C\|v\|_{H^\varepsilon}$, for all $v \in H^\varepsilon$. For getting the *a priori* estimates (2.6) and (2.7), we take $v = u^\varepsilon$ as a test function in the variational formulation (2.5) and, by the coerciveness of a and the continuity of l, we obtain $\|u^\varepsilon\|_{H^\varepsilon} \leq \sqrt{\varepsilon}C$. Thus, taking into account also the estimates given in Lemma 2.2, we are led to (2.6) and (2.7). ■

3. UNFOLDING OPERATORS FOR THE THIN DOMAIN AND COMPACTNESS RESULTS

We briefly recall the definition and some properties of the unfolding operators to be used for our specific geometry (see [2], [7], [1], [26]). These operators will allow us to simultaneously perform homogenization and dimension reduction.

For $x \in \mathbb{R}^3$, we denote by $[x]_Y$ its integer part $\kappa \in \mathbb{Z}^3$, such that $x - [x]_Y \in Y$, and we set $\{x\}_Y = x - [x]_Y$ for $x \in \mathbb{R}^3$. So, for every $x \in \mathbb{R}^3$, we have $x = \varepsilon\left(\left[\frac{x}{\varepsilon}\right]_Y + \left\{\frac{x}{\varepsilon}\right\}_Y\right)$. For $x \in \Omega^\varepsilon$, we have $x = \varepsilon\left(\left[\frac{(\bar{x},0)}{\varepsilon}\right]_Y + \left\{\frac{x}{\varepsilon}\right\}_Y\right)$.

Definition 3.1 *For any Lebesgue measurable function φ in $\Omega^\varepsilon_\alpha$, $\alpha \in \{1,2\}$, we define the periodic unfolding operators by the formula*

$$\mathcal{T}^\varepsilon_\alpha(\varphi)(\bar{x},y) = \varphi\left(\varepsilon\left[\frac{(\bar{x},0)}{\varepsilon}\right]_Y + \varepsilon y\right), \quad \textit{for a.e. } (\bar{x},y) \in \omega \times Y_\alpha.$$

If φ is a function defined in Ω^ε, for simplicity, we write $\mathcal{T}^\varepsilon_\alpha(\varphi)$ instead of $\mathcal{T}^\varepsilon_\alpha(\varphi|_{\Omega^\varepsilon_\alpha})$.

For any function φ which is Lebesgue-measurable on Γ^ε, the periodic boundary unfolding operator $\mathcal{T}^\varepsilon_b$ is defined by

$$\mathcal{T}^\varepsilon_b(\varphi)(\bar{x},y) = \varphi\left(\varepsilon\left[\frac{(\bar{x},0)}{\varepsilon}\right]_Y + \varepsilon y\right), \quad \textit{for a.e. } (\bar{x},y) \in \omega \times \Gamma.$$

Proposition 3.2 *The unfolding operators $\mathcal{T}^\varepsilon_\alpha$ are linear and continuous from $L^2(\Omega^\varepsilon_\alpha)$ to $L^2(\omega \times Y_\alpha)$ and, if φ and ψ are two Lebesgue measurable functions in $\Omega^\varepsilon_\alpha$, one has $\mathcal{T}^\varepsilon_\alpha(\varphi\psi) = \mathcal{T}^\varepsilon_\alpha(\varphi)\mathcal{T}^\varepsilon_\alpha(\psi)$. Moreover,*

(i) for every $\varphi \in L^1(\Omega^\varepsilon_\alpha)$, one has

$$\int_{\Omega^\varepsilon_\alpha} \varphi(x)\,\mathrm{d}x = \varepsilon \int_{\omega \times Y_\alpha} \mathcal{T}^\varepsilon_\alpha(\varphi)(\bar{x},y)\,\mathrm{d}\bar{x}\,\mathrm{d}y;$$

(ii) for every $\varphi \in L^2(\Omega^\varepsilon_\alpha)$, it holds

$$\|\mathcal{T}^\varepsilon_\alpha(\varphi)\|_{L^2(\omega \times Y_\alpha)} = \frac{1}{\sqrt{\varepsilon}}\|\varphi\|_{L^2(\Omega^\varepsilon_\alpha)};$$

(iii) for every $\varphi \in H^1(\Omega^\varepsilon_\alpha)$, one has $\nabla_y\left(\mathcal{T}^\varepsilon_\alpha(\varphi)\right) = \varepsilon\mathcal{T}^\varepsilon_\alpha(\nabla\varphi)$ a.e. in $\omega \times Y_\alpha$;

(iv) for every $\varphi \in L^1(\Gamma^\varepsilon)$, one has

$$\int_{\Gamma^\varepsilon} \varphi(x)\,\mathrm{d}\sigma_x = \int_{\omega\times\Gamma} \mathcal{T}_b^\varepsilon(\varphi)(\bar{x},y)\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y;$$

(v) for every $\varphi \in L^2(\Gamma^\varepsilon)$, it holds

$$\|\mathcal{T}_b^\varepsilon(\varphi)\|_{L^2(\omega\times\Gamma)} = \|\varphi\|_{L^2(\Gamma^\varepsilon)}.$$

Remark 3.3 *In Proposition 3.2, we used the fact that the measure of the set $\mathcal{Y}_1 = (0,1)^2$ equals 1.*

Proposition 3.4 *Let $v^\varepsilon = (v_1^\varepsilon, v_2^\varepsilon) \in H^\varepsilon$ be such that*

$$\|v^\varepsilon\|_{H^\varepsilon} \leq \sqrt{\varepsilon} C. \tag{3.1}$$

Then, up to a subsequence, still denoted by ε, there exist $v_1 \in H_0^1(\omega)$, $v_2 \in L^2(\omega)$, $\widehat{v}_1 \in L^2\left(\omega, H^1_{\overline{per}}(Y_1)\right)$, and $\widehat{v}_2 \in L^2\left(\omega, H^1(Y_2)\right)$ such that

$$\begin{aligned}
&\mathcal{T}_1^\varepsilon(v_1^\varepsilon) \rightharpoonup v_1 \text{ weakly in } L^2\left(\omega, H^1(Y_1)\right),\\
&\mathcal{T}_1^\varepsilon(\nabla_{\bar{x}} v_1^\varepsilon) \rightharpoonup \nabla_{\bar{x}} v_1 + \nabla_{\bar{y}} \widehat{v}_1 \text{ weakly in } L^2(\omega\times Y_1),\\
&\mathcal{T}_1^\varepsilon(\partial_{x_3} v_1^\varepsilon) \rightharpoonup \partial_{y_3} \widehat{v}_1 \text{ weakly in } L^2(\omega\times Y_1),\\
&\mathcal{T}_2^\varepsilon(v_2^\varepsilon) \rightharpoonup v_2 \text{ weakly in } L^2(\omega, H^1(Y_2)),\\
&\mathcal{T}_2^\varepsilon(\nabla v_2^\varepsilon) \rightharpoonup \nabla_y \widehat{v}_2 \text{ weakly in } L^2\left(\omega\times Y_2\right),
\end{aligned}$$

where $\mathcal{M}_\Gamma(\widehat{v}_1) = 0$ and $\mathcal{M}_\Gamma(\widehat{v}_2) = 0$ for a.e. $\bar{x} \in \omega$ and $H^1_{\overline{per}}(Y_1)$ is defined by

$$H^1_{\overline{per}}(Y_1) = \{v \in H^1(Y_1) \,|\, v \text{ is 1-periodic in } y_1 \text{ and } y_2\}.$$

Proof. From (3.1), Lemma 2.2, and the definition of the norm in H^ε, we get

$$\frac{1}{\sqrt{\varepsilon}}\|v_\alpha^\varepsilon\|_{L^2(\Omega_\alpha^\varepsilon)} + \frac{1}{\sqrt{\varepsilon}}\|\nabla v_\alpha^\varepsilon\|_{L^2(\Omega_\alpha^\varepsilon)} \leq C, \qquad \alpha \in \{1,2\}.$$

From Definition 3.1 and Proposition 3.2, (ii), (iii), it follows that there exists a constant $C > 0$, independent of ε, such that

$$\|\mathcal{T}_\alpha^\varepsilon(v_\alpha^\varepsilon)\|_{L^2(\omega\times Y_\alpha)} \leq C, \qquad \|\mathcal{T}_\alpha^\varepsilon(\nabla v_\alpha^\varepsilon)\|_{L^2(\omega\times Y_\alpha)} \leq C, \qquad \alpha \in \{1,2\}.$$

Then, the first three convergences follow by using Proposition 4.4 and Proposition 4.7 in [7] (see, also, [9] and [1]). The last two convergences are an adaptation of Theorems 2.18 and 2.19 in [26] to the present geometry. ▪

4. HOMOGENIZATION RESULTS

By using the periodic unfolding operators and the general compactness results given in the previous section, we shall pass to the limit, with $\varepsilon \to 0$, in the variational formulation (2.5) of problem (2.2). From the *a priori* estimates (2.6)-(2.7) and the compactness results from Proposition 3.4, it follows that there exist $u_1 \in H_0^1(\omega)$, $u_2 \in L^2(\omega)$, $\widehat{u}_1 \in L^2(\omega, H^1_{\mathrm{per}}(Y_1))$, $\widehat{u}_2 \in L^2(\omega, H^1(Y_2))$, with $\mathcal{M}_\Gamma(\widehat{u}_1) = 0$, $\mathcal{M}_\Gamma(\widehat{u}_2) = 0$ and such that, up to a subsequence, for $\varepsilon \to 0$, we get:

$$\begin{aligned}
&\mathcal{T}_1^\varepsilon(u_1^\varepsilon) \rightharpoonup u_1 \quad \text{weakly in } L^2(\omega, H^1(Y_1)),\\
&\mathcal{T}_1^\varepsilon(\nabla_{\bar{x}} u_1^\varepsilon) \rightharpoonup \nabla_{\bar{x}} u_1 + \nabla_{\bar{y}} \widehat{u}_1 \quad \text{weakly in } L^2(\omega \times Y_1),\\
&\mathcal{T}_1^\varepsilon(\partial_{x_3} u_1^\varepsilon) \rightharpoonup \partial_{y_3} \widehat{u}_1 \quad \text{weakly in } L^2(\omega \times Y_1),\\
&\mathcal{T}_2^\varepsilon(u_2^\varepsilon) \rightharpoonup u_2 \quad \text{weakly in } L^2(\omega, H^1(Y_2)),\\
&\mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon) \rightharpoonup \nabla_y \widehat{u}_2 \quad \text{weakly in } L^2(\omega \times Y_2).
\end{aligned} \tag{4.1}$$

As in [34], we introduce the following notation: to every $w = w(\bar{x}) \in H^1(\omega)$, whose gradient $\nabla_{\bar{x}} w(\bar{x})$ has two components, we associate the tridimensional vector $\overline{\nabla} w(\bar{x})$ defined by

$$\overline{\nabla} w(\bar{x}) = (\nabla_{\bar{x}} w(\bar{x}), 0).$$

Theorem 4.1 *Let $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ be the unique solution of the variational problem (2.5). Then, the functions $u_1 \in H_0^1(\omega)$, $u_2 \in L^2(\omega)$, $\widehat{u}_1 \in L^2(\omega, H^1_{per}(Y_1))$, and $\widehat{u}_2 \in L^2(\omega, H^1(Y_2))$ appearing in* (4.1) *are such that $\nabla_{\bar{y}} \widehat{u}_2 = 0$, the pair $(u_1, \widehat{u}_1)$ is the unique solution of the unfolded limit problem*

$$\begin{gathered}
\int_{\omega \times Y_1} A(y)(\overline{\nabla} u_1 + \nabla_y \widehat{u}_1) \cdot (\overline{\nabla} \varphi + \nabla_y \Phi)\, \mathrm{d}\bar{x}\, \mathrm{d}y = \int_\omega f(\bar{x}) \varphi(\bar{x})\, \mathrm{d}\bar{x} + \\
|\Gamma| \mathcal{M}_\Gamma(g) \int_\omega \varphi(\bar{x})\, \mathrm{d}\bar{x} + \int_\omega k_+(\bar{x}) \varphi(\bar{x})\, \mathrm{d}\bar{x} + \int_\omega k_-(\bar{x}) \varphi(\bar{x})\, \mathrm{d}\bar{x},
\end{gathered} \tag{4.2}$$

for all $\varphi \in H_0^1(\omega)$ and $\Phi \in L^2(\omega, H^1_{per}(Y_1))$, and

$$u_2(\bar{x}) = u_1(\bar{x}) + \frac{|Y_2|}{|\Gamma| \mathcal{M}_\Gamma(h)} f(\bar{x}). \tag{4.3}$$

Proof. We start by unfolding the variational formulation (2.5). We get

$$\begin{gathered}
\varepsilon \int_{\omega \times Y_1} \mathcal{T}_1^\varepsilon(A^\varepsilon) \mathcal{T}_1^\varepsilon(\nabla u_1^\varepsilon) \cdot \mathcal{T}_1^\varepsilon(\nabla v_1)\, \mathrm{d}\bar{x}\, \mathrm{d}y + \\
\varepsilon \int_{\omega \times Y_2} \mathcal{T}_2^\varepsilon(A^\varepsilon) \mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon) \cdot \mathcal{T}_2^\varepsilon(\nabla v_2)\, \mathrm{d}\bar{x}\, \mathrm{d}y + \\
\varepsilon \int_{\omega \times \Gamma} h(y)(\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - \mathcal{T}_2^\varepsilon(u_2^\varepsilon))(\mathcal{T}_1^\varepsilon(v_1) - \mathcal{T}_2^\varepsilon(v_2))\, \mathrm{d}\bar{x}\, \mathrm{d}\sigma_y = \\
\varepsilon \int_{\omega \times Y_1} \mathcal{T}_1^\varepsilon(f) \mathcal{T}_1^\varepsilon(v_1)\, \mathrm{d}\bar{x}\, \mathrm{d}y + \varepsilon \int_{\omega \times Y_2} \mathcal{T}_2^\varepsilon(f) \mathcal{T}_2^\varepsilon(v_2)\, \mathrm{d}\bar{x}\, \mathrm{d}y + \\
\varepsilon \int_{\omega \times \Gamma} \mathcal{T}_b^\varepsilon(g^\varepsilon) \mathcal{T}_1^\varepsilon(v_1)\, \mathrm{d}\bar{x}\, \mathrm{d}\sigma_y + \\
\varepsilon \int_{\omega \times \mathcal{Y}_1} \mathcal{T}^\varepsilon(k_+)(\bar{x}, \bar{y}) \mathcal{T}_1^\varepsilon(v_1)(\bar{x}, \bar{y}, \varepsilon)\, \mathrm{d}\bar{x}\, \mathrm{d}\bar{y} +
\end{gathered}$$

$$\varepsilon \int_{\omega\times\mathcal{Y}_1} \mathcal{T}^\varepsilon(k_-)(\bar{x},\bar{y})\mathcal{T}_1^\varepsilon(v_1)(\bar{x},\bar{y},0)\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{y}, \tag{4.4}$$

where $\mathcal{Y}_1 = (0,1)^2$, $\bar{y} = (y_1,y_2)$ and $\mathcal{T}^\varepsilon$ is the classical unfolding operator from [1], Definition 1.2, considered here as an operator from $L^2(\omega)$ to $L^2(\omega\times\mathcal{Y}_1)$.

Recalling that $x = (\bar{x},x_3)$, we choose in the unfolded problem (4.4), as in [30, 31], the test functions

$$v_1 = 0, \quad v_2 = \varepsilon\omega_2(\bar{x})\,\psi_2\left(\left\{\frac{\bar{x}}{\varepsilon}\right\},\frac{x_3}{\varepsilon}\right),$$

with $\omega_2 \in \mathcal{D}(\omega)$, $\psi_2 \in H^1(Y_2)$ (we assume that, for $\alpha\in\{1,2\}$, any function defined on Y_α is extended by $Y-$ periodicity to the whole of $\mathbb{R}^3$). Dividing (4.4) by ε and passing to the limit, we obtain, by using the density of $\mathcal{D}(\omega)\otimes H^1(Y_2)$ in $L^2(\omega,H^1(Y_2))$,

$$\int_{\omega\times Y_2} A(y)\nabla_y\widehat{u}_2\cdot\nabla_y\Phi_2\,\mathrm{d}\bar{x}\,\mathrm{d}y = 0, \quad \forall\Phi_2\in L^2(\omega,H^1(Y_2)), \tag{4.5}$$

which, as in [12, 26, 30, 31], leads to $\nabla_y\widehat{u}_2 = 0$.

We take now in the unfolded problem (4.4) the test functions

$$v_\alpha(\bar{x},x_3) = \varphi_\alpha(\bar{x}) + \varepsilon\omega_\alpha(\bar{x})\,\psi_\alpha\left(\left\{\frac{\bar{x}}{\varepsilon}\right\},\frac{x_3}{\varepsilon}\right), \quad \alpha\in\{1,2\}, \tag{4.6}$$

with $\varphi_\alpha,\omega_\alpha\in\mathcal{D}(\omega)$, $\psi_1\in H^1_{\overline{\text{per}}}(Y_1)$, $\psi_2\in H^1(Y_2)$. One has, for $\alpha\in\{1,2\}$, the following convergences:

$$\mathcal{T}_\alpha^\varepsilon(v_\alpha)\to\varphi_\alpha(\bar{x}) \quad \text{strongly in } L^2(\omega\times Y_\alpha), \tag{4.7}$$

$$\mathcal{T}_\alpha^\varepsilon(\nabla v_\alpha)\to\overline{\nabla}\varphi_\alpha(\bar{x})+\nabla_y\Phi_\alpha \quad \text{strongly in } L^2(\omega\times Y_\alpha), \tag{4.8}$$

where $\Phi_\alpha(\bar{x},y) = \omega_\alpha(\bar{x})\,\psi_\alpha(y)$.

After dividing equation (4.4) by ε, we pass to the limit with $\varepsilon\to 0$, by using convergences (4.1) and (4.7)-(4.8). We detail the passage to the limit only for the terms needing more attention:

$$\int_{\omega\times\Gamma}\mathcal{T}_b^\varepsilon(g^\varepsilon)\mathcal{T}_1^\varepsilon(v_1)\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y =$$

$$\int_{\omega\times\Gamma} g(y)\mathcal{T}_1^\varepsilon(\varphi_1)(\bar{x},y)\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y + \varepsilon\int_{\omega\times\Gamma} g(y)\mathcal{T}_1^\varepsilon(\omega_1)(\bar{x},y)\mathcal{T}_1^\varepsilon(\psi_1)(\bar{x},y)\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y \to$$

$$|\Gamma|\mathcal{M}_\Gamma(g)\int_\omega\varphi_1(\bar{x})\,\mathrm{d}\bar{x},$$

$$\int_{\omega\times\mathcal{Y}_1}\mathcal{T}^\varepsilon(k_+)(\bar{x},\bar{y})\mathcal{T}_1^\varepsilon(v_1)(\bar{x},\bar{y},\varepsilon)\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{y}\to\int_{\omega\times\mathcal{Y}_1}k_+(\bar{x})\varphi_1(\bar{x})\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{y} =$$

$$\int_\omega k_+(\bar{x})\varphi_1(\bar{x})\,\mathrm{d}\bar{x},$$

$$\int_{\omega\times\mathcal{Y}_1}\mathcal{T}^\varepsilon(k_-)(\bar{x},\bar{y})\mathcal{T}_1^\varepsilon(v_1)(\bar{x},\bar{y},0)\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{y}\to\int_\omega k_-(\bar{x})\varphi_1(\bar{x})\,\mathrm{d}\bar{x},$$

by using **(A5)**, (4.6) and the fact that $|\mathcal{Y}_1|=1$. By the density of $\mathcal{D}(\omega)\otimes H^1_{\text{per}}(Y_1)$ in $L^2(\omega,H^1_{\text{per}}(Y_1))$ and of $\mathcal{D}(\omega)\otimes H^1(Y_2)$ in $L^2(\omega,H^1(Y_2))$, we find

$$\int_{\omega\times Y_1}A(y)(\overline{\nabla}u_1+\nabla_y\widehat{u}_1)\cdot(\overline{\nabla}\varphi_1+\nabla_y\Phi_1)\,\mathrm{d}\bar{x}\,\mathrm{d}y+$$

$$|\Gamma|\mathcal{M}_\Gamma(h)\int_\omega(u_1-u_2)(\varphi_1-\varphi_2)\,\mathrm{d}\bar{x}=$$

$$\int_{\omega\times Y_1}f(\bar{x})\varphi_1(\bar{x})\,\mathrm{d}\bar{x}\,\mathrm{d}y+\int_{\omega\times Y_2}f(\bar{x})\varphi_2(\bar{x})\,\mathrm{d}\bar{x}\,\mathrm{d}y+|\Gamma|\mathcal{M}_\Gamma(g)\int_\omega\varphi_1(\bar{x})\,\mathrm{d}\bar{x}+$$

$$\int_\omega k_+(\bar{x})\varphi_1(\bar{x})\,\mathrm{d}\bar{x}+\int_\omega k_-(\bar{x})\varphi_1(\bar{x})\,\mathrm{d}\bar{x},\tag{4.9}$$

We choose now $\varphi_1=0$ and $\Phi_1=0$ in (4.9) and we obtain

$$-|\Gamma|\mathcal{M}_\Gamma(h)(u_1-u_2)=|Y_2|\,f,\ \text{in }\omega,\tag{4.10}$$

which implies (4.3). Replacing (4.10) in (4.9), we obtain, by standard density arguments, (4.2).

The well-posedness of problem (4.2) is a consequence of the Lax-Milgram theorem. Since the solution $(u_1,\widehat{u}_1)$ is unique, the above convergences are valid for the whole sequence. ▪

Theorem 4.2 *Let $(u_1,\widehat{u}_1)$ be the unique solution of the unfolded limit problem (4.2). Then, u_1 satisfies the homogenized problem*

$$\begin{cases}-div_{\bar{x}}\,(A^h\nabla_{\bar{x}}u_1(\bar{x}))=f(\bar{x})+|\Gamma|\mathcal{M}_\Gamma(g)+k_+(\bar{x})+k_-(\bar{x}) & in\ \omega,\\ u_1=0 \quad on\ \partial\omega\end{cases}\tag{4.11}$$

and

$$\widehat{u}_1(\bar{x},y)=-\sum_{j=1}^{2}\frac{\partial u_1}{\partial x_j}(\bar{x})\chi_1^j(y)\quad in\ \omega\times Y_1.\tag{4.12}$$

Here, A^h is the constant homogenized 2×2 matrix whose entries are defined, for $i,j\in\{1,2\}$, by

$$A^h_{ij}=\int_{Y_1}\left(a_{ij}-\sum_{k=1}^{3}a_{ik}\frac{\partial\chi_1^j}{\partial y_k}\right)\mathrm{d}y.\tag{4.13}$$

The function $\chi_1=(\chi_1^1,\chi_1^2)\in(H^1_{per}(Y_1))^2$ is the weak solutions of the cell problem:

$$\begin{cases}-div_y(A(y)(\nabla_y\chi_1^j-e_j))=0 & in\ Y_1,\\ (A(y)(\nabla_y\chi_1^j-e_j))\cdot n=0 & on\ \Gamma,\\ (A(y)(\nabla_y\chi_1^j-e_j))\cdot\nu_\pm=0 & on\ \Sigma^1_\pm,\\ \mathcal{M}_\Gamma(\chi_1^j)=0,\end{cases}\tag{4.14}$$

where n denotes the unit outward normal to Y_2 and $\nu_\pm = (0,0,\pm 1)$.

Proof. By taking $\varphi = 0$ in the unfolded limit problem (4.2), we get:

$$\int_{\omega\times Y_1} A(y)(\overline{\nabla} u_1 + \nabla_y \widehat{u}_1)\cdot\nabla_y \Phi \,\mathrm{d}\bar{x}\,\mathrm{d}y = 0, \tag{4.15}$$

for all $\Phi \in L^2(\omega, H^1_{\text{per}}(Y_1))$. Following a standard procedure, i.e. taking in (4.15) suitable test functions $\Phi \in L^2(\omega, H^1_{\text{per}}(Y_1))$, we are led to

$$\begin{cases} -\mathrm{div}_y(A(y)\nabla_y\widehat{u}_1) = \mathrm{div}_y(A(y)\overline{\nabla} u_1) & \text{in } \omega\times Y_1,\\ (A(y)\nabla_y\widehat{u}_1)\cdot n = -(A(y)\overline{\nabla} u_1)\cdot n & \text{on } \omega\times\Gamma,\\ (A(y)\nabla_y\widehat{u}_1)\cdot \nu_+ = -(A(y)\overline{\nabla} u_1)\cdot \nu_+ & \text{on } \omega\times\Sigma^1_+,\\ (A(y)\nabla_y\widehat{u}_1)\cdot \nu_- = -(A(y)\overline{\nabla} u_1)\cdot \nu_- & \text{on } \omega\times\Sigma^1_-. \end{cases} \tag{4.16}$$

The linearity of the problem (4.16) suggests us to search $\widehat{u}_1(\bar{x},y) = -\overline{\nabla} u_1(\bar{x})\cdot \chi_1(y)$, where the vector $\chi_1(y) = (\chi_1^1(y), \chi_1^2(y), \chi_1^3(y))$ belonging to $(H^1_{\text{per}}(Y_1))^3$ has to be determined. Recalling that $\overline{\nabla} u_1(\bar{x}) = (\nabla_{\bar{x}} u_1(\bar{x}), 0)$, we notice that only the first two components of χ_1 will play a role in our analysis. Inserting this factorization into the equation, we therefore obtain the local problem (4.14). Standard computations then lead to the homogenized problem (4.11), with the constant coefficients given by (4.13). ▪

Remark 4.3 *We notice that the limit problems* (4.11) *and* (4.3) *satisfied by* (u_1, u_2) *can be written as a coupled system, formed by a partial differential equation and an algebraic one and which can be seen as a modified stationary Barenblatt model (see [35]):*

$$\begin{cases} -div_{\bar{x}}(A^h \nabla_{\bar{x}} u_1) + |\Gamma|\mathcal{M}_\Gamma(h)(u_1 - u_2) = f + |\Gamma|\mathcal{M}_\Gamma(g) + k_+ + k_- & \text{in } \omega,\\ -|\Gamma|\mathcal{M}_\Gamma(h)(u_1-u_2) = |Y_2| f & \text{in } \omega,\\ u_1 = 0 & \text{on } \partial\omega. \end{cases}$$

Remark 4.4 *The homogenized matrix A^h is only of dimension two, but some information coming from the vertical direction of the microscopic problem is still present in it. The constant coefficients* (4.13) *are influenced by the third local variable y_3, through the solution χ_1 of the cell problem* (4.14).

5. CONCLUSIONS

A diffusion problem in a thin periodic material formed by two constituents separated by an imperfect interface was analyzed by using the periodic unfolding method, adapted to thin domains. The limit problem is described by a lower-dimensional modified Barenblatt system.

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