LOCALIZED EXCITATIONS AND THEIR COLLISIONAL DYNAMICS IN (2+1)-DIMENSIONAL BROER-KAUP-KUPERSHMIDT EQUATION

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In this paper, we investigate the (2+1)-dimensional Broer-Kaup-Kupershmidt equation employing the truncated Painlevé approach and we generate localized solutions like dromions, lumps, and rogue waves. The collisional dynamics of the localized solutions is then studied. While the dromions are found to undergo inelastic collision exchanging energy among them, the lumps are found to be noninteracting. We also bring out the unstable nature of rogue waves.

Key words: Trunctated Painlevé approach, Dromions, Rogue waves, Lumps.

1. INTRODUCTION

The revival of interest in the investigation of (2+1)-dimensional nonlinear partial differential equationa (PDEs) can be attributed to the identification of rogue waves [1–10]. In general, rogue waves occur in deep ocean. The most important characteristic feature of rogue waves is that they appear from nowhere and disappear without a trace. In recent times, rogue waves have been identified in various fields such as hydrodynamics [11], nonlinear optics [12], Bose-Einstein condensates [13], and plasma physics [14]. Dromions [15–20], which are exponentially localized solutions [21–26], are another interesting classes of solutions in (2+1) dimensional integrable systems. They essentially arise by virtue of coupling of the field variable to a potential. In addition to rogue waves and dromions, there exist lumps that are algebraically decaying solutions in (2+1)-dimensional nonlinear PDEs [27–31]. While dromions have been known to undergo both elastic and inelastic collisions, lumps do not interact at all. On the other hand, rogue waves are found to be highly unstable. In addition to the above, (2+1)-dimensional nonlinear PDEs also admit traveling wave solutions [32], breathers [33, 34], etc. the The contrasting nature of these localized solutions makes the investigation of (2+1) nonlinear PDEs an interesting journey.

In this paper, we investigate the (2+1)-dimensional Broer-Kaup-Kupershmidt (BKK) system [35] employing the truncated Painlevé appproach and generate localized solutions such as dromions, rogue waves, and lumps. We also study their collisional dynamics. The present paper is structured as follows. In Sec. 2, we show how the (2+1)-dimensional BKK equation can be solved using the truncated Painlevé approach [36–43] and the solution is obtained in terms of lower dimensional arbitrary functions. Section 3 is devoted to the construction of various localized coherent excitations by suitably harnessing the lower dimensional arbitrary functions of space and time. Finally, in Sec. 4, we summarize our results.

2. TRUNCATED PAINLEVÉ APPROACH

We now consider the (2+1)-dimensional BKK system in the following form [35]:

$$H_{ty} - H_{xxy} + 2H_y H_x + 2H H_{xy} + 2G_{xx} = 0, \tag{1}$$

$$G_t + G_{xx} + 2H_x G + 2H G_x = 0. \tag{2}$$

We now effect a local Laurent expansion in the neighbourhood of a noncharacteristic singular manifold $\phi(x,y,t) = 0$, $\phi_x \neq 0$, $\phi_y \neq 0$. Assuming the leading orders of the solutions of the Eqs. (1) and (2) to have the form

$$H = H_0 \phi^{\alpha}, \quad G = G_0 \phi^{\beta}, \tag{3}$$

where H_0 and G_0 are analytic functions of (x,y,t) and α and β are integers to be determined, we now substitute Eq. (3) into Eqs. (1) and (2) and balance the most dominant terms to obtain

$$\alpha = -1, \beta = -2 \tag{4}$$

with the condition

$$H_0 = \phi_x, G_0 = -\phi_x \phi_y. \tag{5}$$

By truncating the Laurent series of the solutions of Eqs. (1) and (2) at the constant level term, one obtains the following Bäcklund transformation

$$H = \frac{H_0}{\phi} + H_1, \tag{6}$$

$$G = \frac{G_0}{\phi^2} + \frac{G_1}{\phi} + G_2. \tag{7}$$

Assuming the following seed solution

$$H_1 = H_1(x,t), \quad G_2 = 0 \tag{8}$$

we now substitute Eq. (6) and Eq. (7) with the above seed solution given by Eq. (8) into Eqs. (1) and (2) and collecting the coefficients of (ϕ^{-4}, ϕ^{-4}) we obtain

$$6H_0 \phi_x^2 \phi_y + 6H_0^2 \phi_x \phi_y + 12 G_0 \phi_x^2 = 0, \tag{9}$$

$$6G_0 \phi_x^2 - 6G_0 H_0 \phi_x = 0. \tag{10}$$

From Eqs. (9) and (10), we obtain

$$H_0 = \phi_x \tag{11}$$

and

$$G_0 = -\phi_x \phi_y. \tag{12}$$

Collecting the coefficients of (ϕ^{-3}, ϕ^{-3}), we have

$$2H_0\phi_y\phi_t - 4H_0H_{0x}\phi_y - 4H_0H_{0y}\phi_x + 4H_0H_1\phi_x\phi_y - 8G_{0x}\phi_x - 4H_{0x}\phi_x\phi_y$$
$$+4G_1\phi_x^2 - 2H_{0y}\phi_x^2 - 2H_0^2\phi_{xy} - 4H_0\phi_{xy}\phi_x - 4G_0\phi_{xx} - 2H_0\phi_{xx}\phi_y = 0, \tag{13}$$
$$-2G_0\phi_t - 4G_0\phi_x + 2G_1\phi_x^2 - 2G_0\phi_{xx} + 2H_{0x}G_0$$
$$-4H_0G_1\phi_x - 2H_0G_{0x} - 4G_0H_1\phi_x = 0. \tag{14}$$

Substituting Eqs. (11) and (12) in Eqs. (13) and (14), we get

$$H_1 = \frac{(-\phi_t + \phi_{xx})}{2\phi_x}, \tag{15}$$

$$G_1 = \phi_{xy}. \tag{16}$$

Collecting the coefficients of (ϕ^{-2}, ϕ^{-2}), we have

$$H_{0y}\phi_t - 2H_0H_{1x}\phi_y - H_{0t}\phi_y - H_0\phi_{ty} + H_{0xx}\phi_y + 2H_{0xy}\phi_x + 2H_{0x}\phi_{xy} + H_{0y}\phi_{xx}$$
$$+H_0\phi_{xxy} + 2H_{0x}H_{0y} + 2H_0H_{0xy} + 2G_{0xx} - 4G_{1x}\phi_x - 2G_1\phi_{xx} - 2H_1\phi_yH_{0x}$$
$$-2H_0H_1\phi_{xy} - 2H_1H_{0y}\phi_x = 0, \tag{17}$$
$$G_{0t} - G_1\phi_t - 2G_{1x}\phi_x + G_{0xx} - G_1\phi_{xx} + 2H_{0x}G_1 + 2H_0G_{1x} + 2H_{1x}G_0$$
$$+2H_1G_{0x} - 2G_1H_1\phi_x = 0. \tag{18}$$

Substituting Eqs. (11), (12), (15), and (16) in Eqs. (17) and (18), we obtain

$$\phi_{xy}(\phi_{xx} + \phi_t) - \phi_x(\phi_{yt} + \phi_{xxy}) = 0. \tag{19}$$

The structure of the trilinear equation (19) suggests that the manifold can be partitioned as

$$\phi = \phi_1(x,t) + \phi_2(y), \tag{20}$$

where $\phi_1(x,t)$, $\phi_2(y)$ are arbitrary functions in the indicated variables. Substituting Eq. (20) in (11), (12), (15), and (16) leads to

$$H_0 = \phi_{1x}, \tag{21}$$

$$G_0 = -\phi_{1x}\phi_{2y}, \tag{22}$$

$$H_1 = \frac{-\phi_{1t} + \phi_{1xx}}{2\phi_{1x}}, \tag{23}$$

$$G_1 = 0. \tag{24}$$

Again, collecting the coefficients of (ϕ^{-1}, ϕ^{-1}), we have

$$H_{1x}H_{0y} + H_{0yt} + 2H_1H_{0xy} + 2G_{1xx} - H_{0xxy} = 0, \tag{25}$$

$$2G_1H_{1x}+G_{1t}+2H_1G_{1x}+G_{1xx}=0. \quad (26)$$

It is obvious that the substitution of H_0, H_1 and G_1 identically satisfy Eq. (25) and Eq. (26). Again, collecting the coefficients of (ϕ^0,ϕ^0), we are left with only one equation

$$H_{1yt}+2H_{1y}H_{1x}+2H_1H_{1xy}-H_{1xxy}=0, \quad (27)$$

which is again an identity. Thus, the (2+1)-dimensional BKK equation (1) - (2) has been solved completely through the truncated Painlevé approach and the closed form of the fields H and G are given by

$$H=\frac{\phi_{1x}}{(\phi_1(x,t)+\phi_2(y))}-\frac{\phi_{1t}+\phi_{1xxx}}{2\phi_{1x}}, \quad (28)$$

$$G=\frac{-\phi_{1x}\phi_{2y}}{(\phi_1(x,t)+\phi_2(y))^2}. \quad (29)$$

3. LOCALIZED SOLUTIONS OF THE (2+1)-DIMENSIONAL BKK EQUATION

3.1. DROMIONS

3.1.1. The (1,1) Dromion Solution

To construct a (1,1) dromion solution using Eq. (29), we choose

$$\phi_1(x,t)=a_1\tanh(b_1x-b_2t+e_1)+g_1, \quad (30)$$

$$\phi_2(y)=c_1\tanh(d_1y+e_2)+g_2. \quad (31)$$

The (1,1) dromion takes the form

$$G=\frac{G_1}{G_2}, \quad (32)$$

where

$$G_1=-a_1b_1c_1d_1\,\mathrm{sech}^2(e_1-b_2t+b_1x)\,\mathrm{sech}^2(e_2+d_1y), \quad (33)$$

$$G_2=(g_1+g_2+a_1\tanh(e_1-b_2t+b_1x)+c_1\tanh(d_1y+e_2))^2. \quad (34)$$

The (1,1) dromion for the parametric choice $a_1=-1, c_1=b_1=b_2=d_1=1, e_1=e_2=0, g_1=3$, and $g_2=2$ is shown in Fig. 1. The (1,1) dromion travels with unit velocity in the $x-y$ plane.

3.1.2. The (2,1) Dromions Solution

To construct the (2,1) dromion solution using Eq. (29), we choose

$$\phi_1(x,t)=a_1\tanh(b_1x-b_2t+e_1)+a_2\tanh(b_3x-b_4t+e_2)+g_1,$$

$$\phi_2(y)=c_1\tanh(d_1y+e_3)+g_2. \quad (35)$$

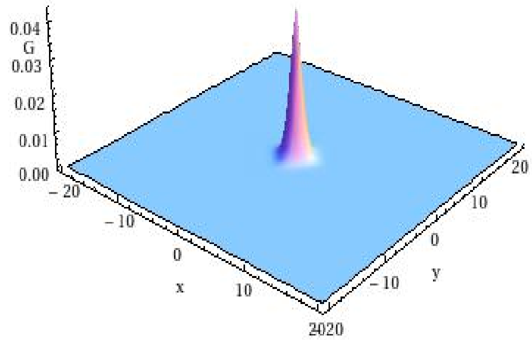

Fig. 1 – The (1,1) dromion solution at $t = 0$.

The (2,1) dromion solution takes the following form

$$G = \frac{(c_1 d_1(-a_1 b_1 \operatorname{sech}^2(u_1) - a_2 b_3 \operatorname{sech}^2(u_2)) \operatorname{sech}^2(u_3))}{(g_1 + g_2 + a_1 \tanh(u_1) + a_2 \tanh(u_2) + c_1 \tanh(u_3))^2}. \quad (36)$$

The (2,1) dromion for the parametric choice $u_1 = e_1 - b_2 t + b_1 x, u_2 = e_2 - b_4 t + b_3 x, u_3 = e_3 + d_1 y, a_1 = -1, a_2 = -1, c_1 = b_1 = b_3 = b_4 = d_1 = 1, e_1 = e_2 = e_3 = 0, b_2 = -1, g_1 = 2$, and $g_2 = 3$ is shown in Fig. 2. The (2,1) dromions are traveling along the x-direction with the same velocity in opposite directions in the $x - y$ plane and undergo inelastic collision by exchanging energy among them.

3.1.3. Asymptotic analysis of the (2,1) dromion solution

Since both the dromions in Figs. 2a and 2c corresponding to the solution (36) are traveling along the x-direction following one another, it is enough to do this analysis for $y = 0$ (and $e_3 = 0$ so that $g_2 = 0$). Similar analysis holds good for any other value of y and $e_3 \neq 0$. This restriction corresponds to the cross section of dromions, which are essentially solitons. We analyze the limits $t \to -\infty$ and $t \to +\infty$ separately so as to understand the interaction of dromions centered around $u_1 \approx 0$ or $u_2 \approx 0$. Without loss of generality, let us assume $b_2 > b_4$ and $b_1 < b_3$. Then, we find in the limit t $\to \pm\infty$, u_1 and u_2 take the following limiting values:

(1) As $t \to -\infty$:

$u_1 \approx 0, u_2 \to -\infty$,

$u_2 \approx 0, u_1 \to +\infty$.

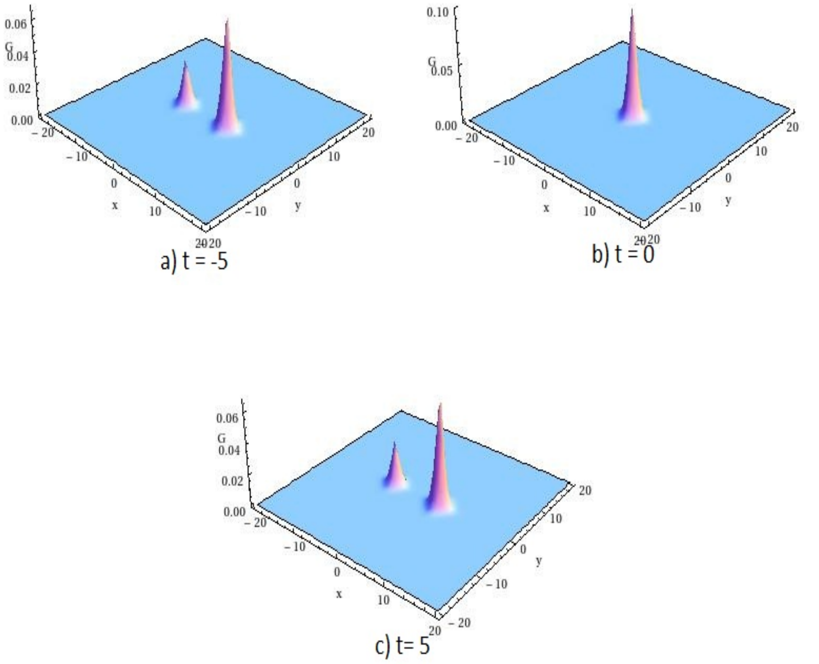

Fig. 2 – The (2,1) dromions.

(2) As $t \rightarrow +\infty$:
$u_1 \approx 0$, $u_2 \rightarrow +\infty$,
$u_2 \approx 0, u_1 \rightarrow -\infty$.

1 Before interaction (as $t \rightarrow -\infty$): For $u_1 \approx 0$, $u_2 \rightarrow -\infty$ and $a_1 = 1; a_2 = -1, a_3 = 1, c_1 = 1; g_2 = 1$, the (2,1) dromion solution (36) becomes (soliton solution corresponding to dromion 1)

$$G = \frac{-b_1 d_1}{g_1(g_1+2)} \operatorname{sech}(u_1+\delta_1)^2, \delta_1 = \frac{1}{2}\log\left[\frac{g_1+2}{g_1}\right]. \tag{37}$$

For $u_2 \approx 0$, $u_1 \rightarrow \infty$ the solutions (36) becomes (soliton solution corresponding to dromion 2)

$$G = \frac{b_3 d_1}{g_1(g_1+2)} \operatorname{sech}(u_2-\delta_1)^2. \tag{38}$$

2 After interaction (as $t \to +\infty$): For $u_1 \approx 0$, $u_2 \to +\infty$ the solutions (36) becomes (soliton solution corresponding to dromion 1)

$$G = \frac{-b_1 d_1}{g_1(g_1-2)} \operatorname{sech}(u_1+\delta_2)^2, \delta_2 = \frac{1}{2}\log\left[\frac{g_1}{g_1-2}\right]. \tag{39}$$

For $u_2 \approx 0$, $u_1 \to -\infty$, the solutions (36) becomes (soliton solution corresponding to dromion 2)

$$G = \frac{-b_3 d_1}{g_1(g_1-2)} \operatorname{sech}(u_2-\delta_2)^2. \tag{40}$$

3.2. LUMPS

3.2.1. The One Lump

To construct the one lump solution using Eq. (29), we choose

$$\phi_1(x,t) = \frac{1}{(1+(e_1x-e_2t-e_3))^2}, \tag{41}$$

$$\phi_2(y) = \frac{1}{(1+(ky-h)^2)}. \tag{42}$$

The one lump solution takes the following form

$$G = \frac{G_3}{G_4G_5}, \tag{43}$$

where

$$G_3 = 4e_1k(-h+ky), \tag{44}$$

$$G_4 = ((1-e_3-e_2t+e_1x)^3(1+(-h+ky)^2)^2, \tag{45}$$

$$G_5 = \left(\frac{1}{(1-e_3-e_2t+e_1x)^2}+\frac{1}{(1+(-h+ky)^2)}\right)^2, \tag{46}$$

where $e_1 = e_2 = k = h = 3$, and $e_3 = 2$. The time evolution of single lump solution is shown in Fig. 3.

3.2.2. The Two Lumps

To construct the two-lumps solution using Eq. (29), we choose

$$\phi_1(x,t) = \frac{1}{(1+(e_1x-e_2t-e_3))^2}+\frac{1}{(1+(e_4x-e_5t-e_6))^2},$$

$$\phi_2(y) = \frac{1}{(1+(ky-h)^2)}+\frac{1}{(1+(k_1y-h_1)^2)}, \tag{47}$$

where $e_1 = e_2 = e_4 = e_5 = k = k_1 = h = 1, e_3 = 2, h_1 = 8$, and $e_6 = 2$. The time evolution of the two-lumps solution is shown in Fig. 4. The two lumps do not interact with each other when they travel in the two-dimensional plane.

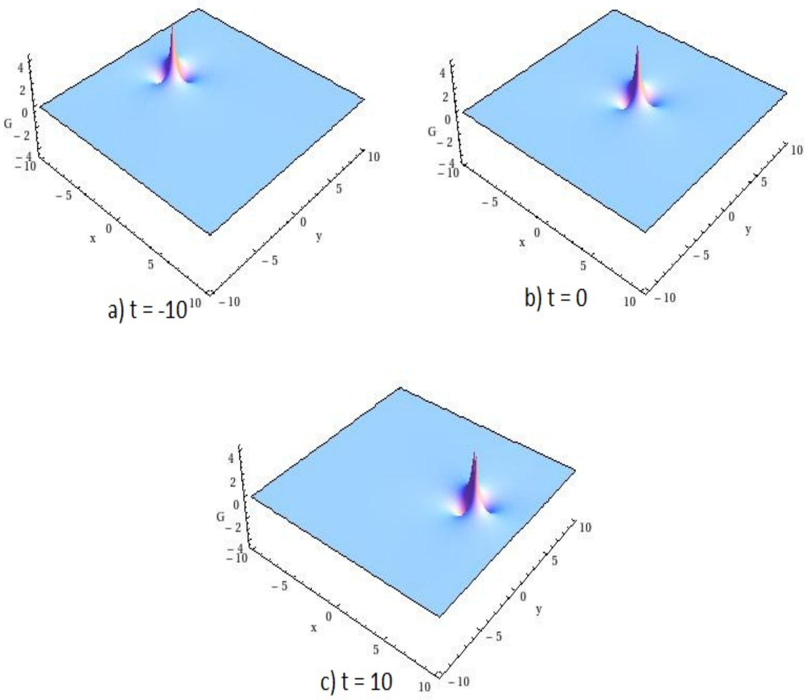

Fig. 3 – Time evolution of the single-lump solution.

3.3. ROGUE WAVES

3.3.1. The Single Rogue Wave

To construct the single rogue-wave solution using Eq. (29), we choose

$$\phi_1(x,t) = \frac{-cx}{(1+a_1x^2+b_1t^2)^2} + \frac{1}{(2b_1)^2}, \tag{48}$$

$$\phi_2(y) = \frac{3hy^4}{(1+(y-\alpha)^2c_1)^2}. \tag{49}$$

The single rogue wave takes the following form

$$G = \frac{G_6}{G_7G_8}, \tag{50}$$

where

$$G_6 = 12c_1hy^3(1+y(y-\alpha)+\alpha^2)(1-3a_1cx^2+b_1t^2), \tag{51}$$

$$G_7 = (1+(y-\alpha)^2c_1)^3(1+a_1x^2+b_1t^2)^3, \tag{52}$$

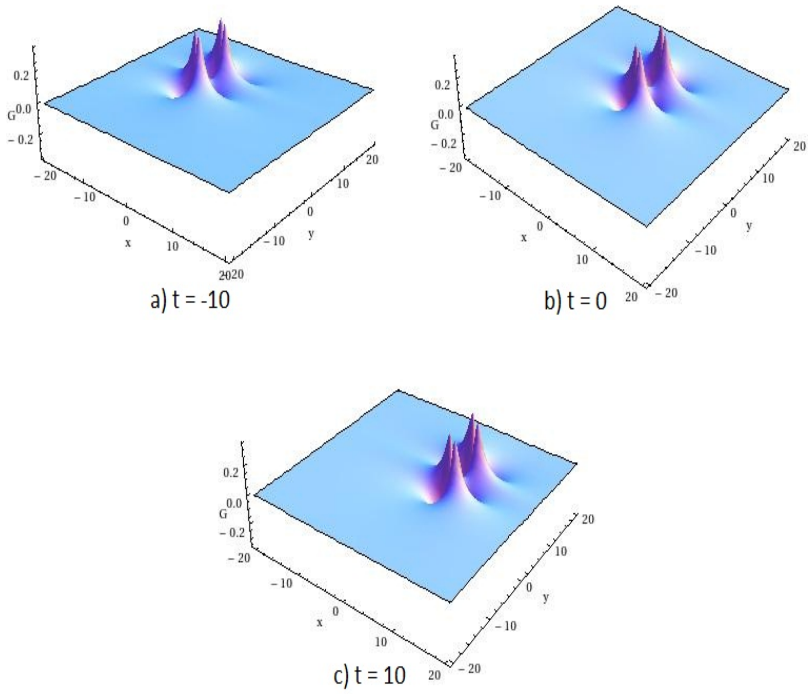

Fig. 4 – Time evolution of the two-lump solution.

$$G_8 = \frac{(3hy^4)}{(1+(y-\alpha)^2 c_1)^2} + \frac{1}{4b_1^2} - \frac{cx}{(1+a_1x^2+b_1t^2)^2}. \tag{53}$$

The time evolution of the rogue wave indicates its unstable nature. The amplitude of the rogue wave is maximum at time $t = 0$ and is shown in Fig. 5 for the parametric choice $a_1 = 0.8, b_1 = 0.05, c_1 = 0.5, c = 35$, $\alpha = 2$, and $h = 45$.

3.3.2. The Two Rogue Waves

To construct the two rogue-waves solution using Eq. (29), we choose

$$\phi_1(x,t) = \left(\frac{-cx}{(1+a_1x^2+b_1t^2)^2} + \frac{1}{(2b_1)^2}\right)^2 + \left(\frac{-c(x-8)}{(1+a_1(x-8)^2+b_1t^2)^2} + \frac{1}{(2b_1)^2}\right)^2,$$

$$\phi_2(y) = \frac{3d_1y^4}{(1+y^2c_1+c_2c_3y)^2} + \frac{3d_1(y-8)^4}{(1+(y-8)^2c_1+c_2c_3(y-8))^2},$$

where $a_1 = 0.8, b_1 = 0.05, c_1 = 0.5, c_2 = 2.5, c_3 = 4, d_1 = 1$, and $c = 35$. The rogue waves are spatially located at two different positions in the $x-y$ plane. It is possible to vary the spatial location of the rogue waves by appropriately varying the para-

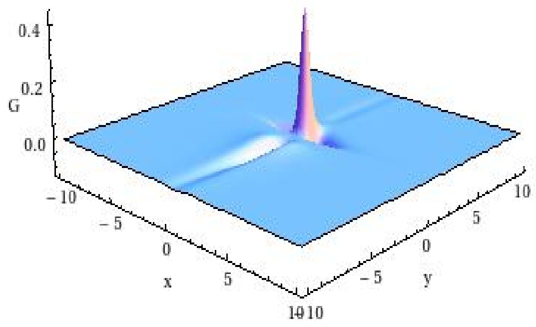

Fig. 5 – The single rogue-wave solution at $t = 0$.

metric choices. The rogue waves have the maximum amplitude at time $t = 0$ and is shown in Fig. 6.

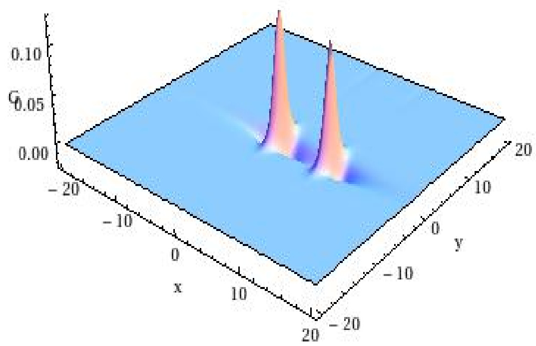

Fig. 6 – Two bright rogue-wave solution at $t = 0$.

4. CONCLUSION

In this paper, we have considered the (2+1)-dimensional Broer-Kaup-Kupershmidt equation and we have investigated it using the truncated Painlevé approach to obtain its solution in closed form involving two lower dimensional arbitrary functions of space and time. The presence of lower dimensional arbitrary functions of space and time enables us to construct localized solutions like dromions, lumps, and rogue waves. Besides, their dynamics is also clearly brought out.

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